

Math 218D-1: Homework #13

due Wednesday, April 17, at 11:59pm

1. For each quadratic form $q(x_1, x_2)$ of HW12#15(a,b), first **i)** draw the solutions of $q(x_1, x_2) = 1$, being sure to draw the shortest and longest solutions, and then **ii)** find the maximum and minimum values of $\|x\|^2$ subject to the constraint $q(x) = 1$, and at which points (x_1, x_2) these values are attained.

What happens if you try to extremize $\|x\|^2$ subject to

$$q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2 = 1?$$

(This is the form from part (c) of HW12#15.)

2. For the quadratic form

$$q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3$$

of HW12#16, find the maximum and minimum values of $\|x\|^2$ subject to the constraint $q(x) = 1$, along with the points (x_1, x_2, x_3) at which these values are attained.

3. **a)** Consider the quadratic form

$$q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3,$$

of HW12#16. Find the *smallest* value of $q(x)$ subject to the constraints $\|x\| = 1$ and $x \perp \frac{1}{3}(1, -2, 2)$. At which vectors x is this minimum attained?

- b)** Consider the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 7x_3^2 - 16x_1x_2 + 8x_1x_3 + 8x_2x_3.$$

of HW12#17. Find the *largest* value of $q(x)$ subject to the constraints $\|x\| = 1$ and $x \perp \frac{1}{\sqrt{5}}(0, 1, 2)$. At which vectors x is this maximum attained?

4. For each matrix A , find the minimum and maximum values of $\|Ax\|^2$ subject to the constraint $\|x\| = 1$. At which vectors are these extrema achieved? Check your work by choosing a unit vector x maximizing $\|Ax\|^2$, computing $b = Ax$, and verifying that $\|b\|^2$ is equal to the maximum.

$$\text{a) } \begin{pmatrix} 3 & -1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \quad \text{c) } \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

5. Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 & 4 & -3 \\ 1 & 7 & -2 & 3 & -5 \\ 2 & 0 & 8 & -1 & 1 \\ 1 & 2 & 0 & 3 & 9 \end{pmatrix}.$$

- a) Find a unit vector u_1 maximizing $\|Ax\|^2$ subject to $\|x\| = 1$.
 - b) Find the maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$ and $x \perp u_1$.
 - c) Find the minimum value of $\|Ax\|$ subject to $\|x\| = 1$ without doing any work.
- You'll need to use a computer algebra system. With the Sage cell on the course webpage, you'd want something like this:

```
A = Matrix([[ 3., 2., -1., 4., -3.],
            [ 1., 7., -2., 3., -5.],
            [ 2., 0., 8., -1., 1.],
            [ 1., 2., 0., 3., 9.]])
pprint((A.T*A).eigenvects())
```

(Entering numbers as “3.” instead of “3” forces SymPy to perform a floating-point computation instead of a symbolic one.)

6. In this problem, we will touch on the role of quadratic optimization in *spectral graph theory*. Spectral graph theory is the study of graphs using linear algebra, and is widely applied to problems in networking and partitioning. (Google’s PageRank algorithm can be formulated as a spectral graph theory problem.)

A *graph* is a set of *vertices*, or points, connected by a set of *edges*. For simplicity, we will assume that each edge has distinct endpoints (i.e., there are no loop edges), and that there is at most one edge connecting any two vertices: such a graph is called *simple*. Under these assumptions, an edge is determined by the two vertices it connects, so we can write $e = (1, 2)$ for the edge connecting vertices 1 and 2. We also write $i \sim j$ if (i, j) is an edge of G . The *degree* of a vertex is the number of edges connected to it; the degree of vertex i is written $\deg(i)$.

Let G be a graph with n vertices labeled $1, 2, \dots, n$. We consider a vector $x \in \mathbf{R}^n$ as a way to assign a real number to each vertex: the i th coordinate x_i is the number attached to the i th vertex. The *Laplacian* of G is the $n \times n$ matrix L whose (i, j) entry is

$$L_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if there is an edge from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise.} \end{cases}$$

Note that L is symmetric. Let $x \in \mathbf{R}^n$ and let $y = Lx$. Then the i th coordinate of y is

$$(\star) \quad y_i = x_i \deg(i) - \sum_{j \sim i} x_j = \sum_{j \sim i} (x_i - x_j).$$

In other words, y is the vector that assigns the number $\sum_{j \sim i} (x_i - x_j)$ to vertex i .

The eigenvalues of the graph Laplacian contain important information about the structure of the graph.

a) Show that the vector $\mathbf{1} = (1, 1, \dots, 1) \in \mathbf{R}^n$ is in the null space of L .

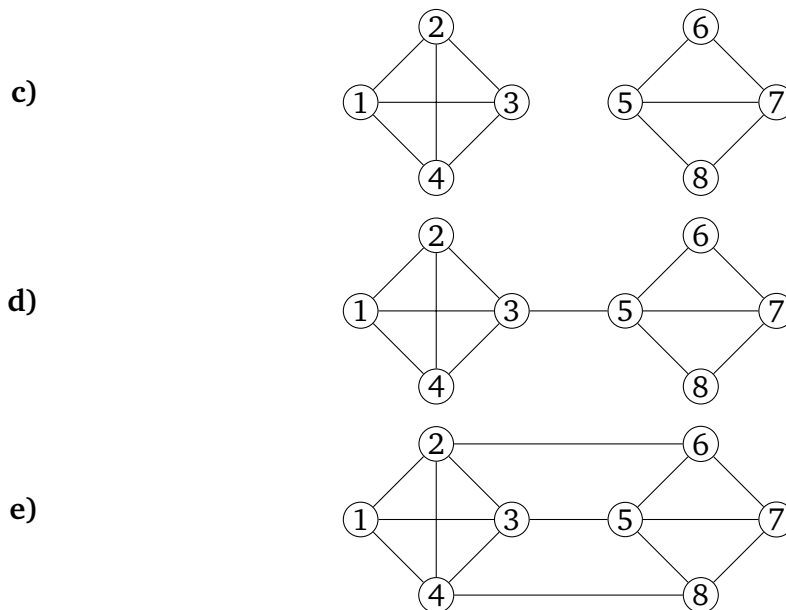
It follows that 0 is always an eigenvalue of L .

b) Show that $x^T Lx = \sum_{j \sim i} (x_i - x_j)^2$. Explain why L is positive-semidefinite.

Since L is positive-semidefinite, all of its eigenvalues are *nonnegative*, so 0 is the smallest eigenvalue of L . The fact that 0 is an eigenvalue gives us no information about the graph, so we wish to “rule it out” by imposing the constraint $x \perp \mathbf{1}$.

According to b), minimizing $q(x) = x^T Lx$ subject to the constraints $\|x\|^2 = 1$ and $x \perp \mathbf{1}$ amounts to finding a way to assign a number to each vertex such that *neighboring vertices have similar values*, but such that the sum of the values is zero ($x \perp \mathbf{1}$) and the sum of their squares is 1 ($\|x\| = 1$).

For each of the following graphs, i) compute the Laplacian matrix L and ii) minimize $x^T Lx$ subject to $x \perp \mathbf{1}$ and $\|x\| = 1$. iii) For a (unit) vector x achieving this minimum, draw the number x_i next to vertex i on the graph. iv) What does the second-smallest eigenvalue say about the graph? (This is open-ended.)



You should feel free to use a computer algebra system to compute the eigenvalues and eigenvectors. For instance, you can use SymPy in the Sage cell on the course webpage. Finding the eigenvalues and eigenvectors of a matrix in SymPy is done as follows: if your matrix is

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

then you would type:

```
A = Matrix([[7.,2.,0.],[2.,6.,2.],[0.,2.,5.]])
pprint(A.eigenvects())
```

(Entering numbers as “3.” instead of “3” forces SymPy to perform a floating-point computation instead of a symbolic one.) The output is a list of tuples of the form (eigenvalue, multiplicity, eigenspace basis)—note that the eigenspace basis will not necessarily be orthonormal.

7. For each matrix A , find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

$$\begin{array}{lll} \text{a) } \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} & \text{b) } \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} & \text{c) } \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix} \\ \text{d) } \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} & \text{e) } \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix} \end{array}$$

8. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 7(a). Let σ_1, σ_2 be the singular values of A . Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

9. Find the matrix A satisfying

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

and write the SVD of A in outer product form.

[Hint: Start by finding the SVD.]

10. Let A be a matrix with nonzero *orthogonal* columns w_1, \dots, w_n of lengths $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$, respectively. Find the SVD of A in outer product form.
11. Let S be a symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_r| > 0 = \lambda_{r+1} = \dots = \lambda_n$. Let $\{v_1, \dots, v_n\}$ be an orthonormal eigenbasis, where v_i has eigenvalue λ_i .
- Show that the singular values of S are $|\lambda_1|, \dots, |\lambda_r|$. In particular, $\text{rank}(S) = r$.
 - Find the singular value decomposition of S in outer product form, in terms of the λ_i and the v_i .

12. a) Show that all singular values of an orthogonal matrix are equal to 1.
- b) Let A be an $m \times n$ matrix, let Q_1 be an $m \times m$ orthogonal matrix, and let Q_2 be an $n \times n$ orthogonal matrix. Show that A has the *same singular values* as Q_1AQ_2 . [Hint: Use HW10#11.]

Remark: This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by **simple orthogonal matrices**.

13. Let A be a matrix of full column rank and let $A = QR$ be the QR decomposition of A .
- a) Show that A and R have the same singular values $\sigma_1, \dots, \sigma_r$ and the same right singular vectors v_1, \dots, v_r .
- b) What is the relationship between the left singular vectors of A and R ?
14. Let A be a matrix with first singular value σ_1 and first right singular vector v_1 . Recall that the *matrix norm* of A is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$, and is denoted $\|A\|$.
- a) Show that $\|Ax\|$ is maximized at $x = v_1$ (subject to $\|x\| = 1$), with maximum value σ_1 .
- b) Suppose now that A is square and λ is an eigenvalue of A . Show that $|\lambda| \leq \sigma_1$. (You may assume λ is real, although it is also true for complex eigenvalues.)

This shows that *the largest singular value is at least as big as the largest eigenvalue*.

15. a) Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- b) Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- c) Note that A is very close to A' numerically. Were the eigenvalues of A close to the eigenvalues of A' ? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are numerically stable*. This is another advantage of the SVD.

- 16.** Decide if each statement is true or false, and explain why.
- a) The left singular vectors of A are eigenvectors of $A^T A$ and the right singular vectors are eigenvectors of AA^T .
 - b) For any matrix A , the matrices AA^T and $A^T A$ have the same eigenvalues.
 - c) If S is symmetric, then the nonzero eigenvalues of S are its singular values.
 - d) If A does not have full column rank, then 0 is a singular value of A .
 - e) Suppose that A is invertible with singular values $\sigma_1, \dots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \dots, \sigma_n + c$.
 - f) The right singular vectors of A are orthogonal to $\text{Nul}(A)$.