Math 218D-1: Homework #14

due Wednesday, April 24, at 11:59pm

1. For each matrix A of HW13#7:

$$\textbf{a)}\ \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

find the singular value decomposition in the matrix form

$$A = U\Sigma V^T.$$

2. For each matrix A of Problem 1, write down orthonormal bases for all four fundamental subspaces. (This can be read off from your answers to Problem 1.)

3. **a)** Let A be an invertible $n \times n$ matrix. Show that the product of the singular values of A equals the absolute value of the product of the (real and complex) eigenvalues of A (counted with algebraic multiplicity).

[**Hint:** Both equal $|\det(A)|$. What is $\det(A^T A)$?]

b) Find an example of a 2×2 matrix A with distinct positive eigenvalues that are not equal to any of the singular values of A.

[**Hint:** One of the matrices in HW13#7 works.]

4. Let S be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $S = QDQ^T$ be an orthogonal diagonalization of S, where D has diagonal entries $\lambda_1, \ldots, \lambda_n$.

Show that $S = QDQ^T$ is a singular value decomposition if and only if S is positive-semidefinite. [See HW13#11.]

5. Let A be a square, invertible matrix with singular values $\sigma_1, \ldots, \sigma_n$.

a) Show that A^{-1} has the same singular vectors as A^T, with singular values $\sigma_n^{-1} \geq \cdots \geq \sigma_1^{-1}$. [**Hint:** invert $A = U\Sigma V^T$.]

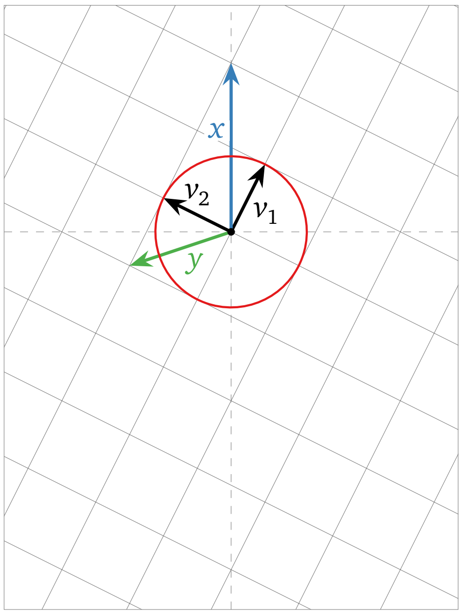
b) Let λ be an eigenvalue of A. Use HW13#14(b) and **a)** to show that $\sigma_n \leq |\lambda|$.

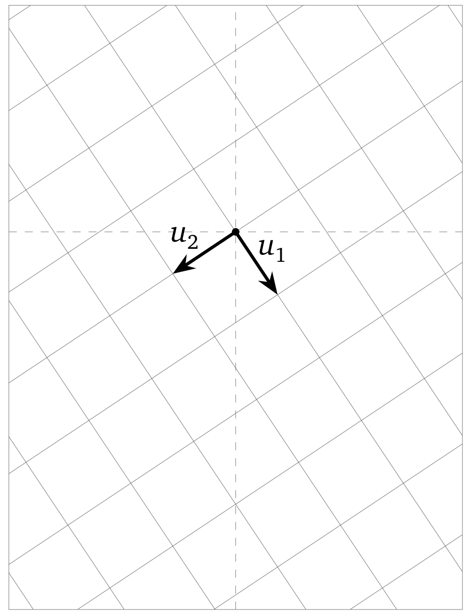
It follows that the absolute values of all eigenvalues of A are contained in the interval $[\sigma_n, \sigma_1]$. Compare Problem 3.

6. A certain 2×2 matrix A has singular values $\sigma_1 = 2$ and $\sigma_2 = 1.5$. The right-singular vectors v_1, v_2 and the left-singular vectors u_1, u_2 are shown in the pictures below.

a) Draw Ax and Ay in the picture on the right.

b) Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit circle by A) in the picture on the right.

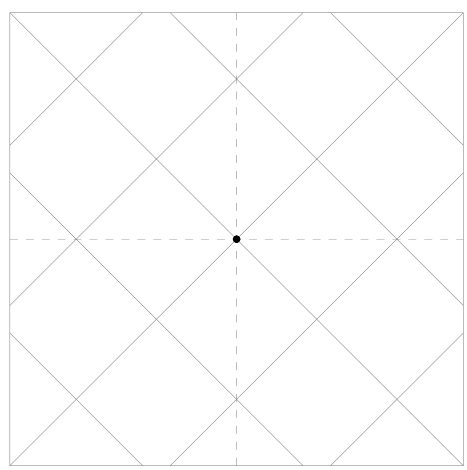




7. Consider the following 3×2 matrix A and its SVD:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^T.$$

Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit sphere by A) in the picture on the right.



8. A certain matrix A has 2 rows and 10 columns. Its SVD has the form

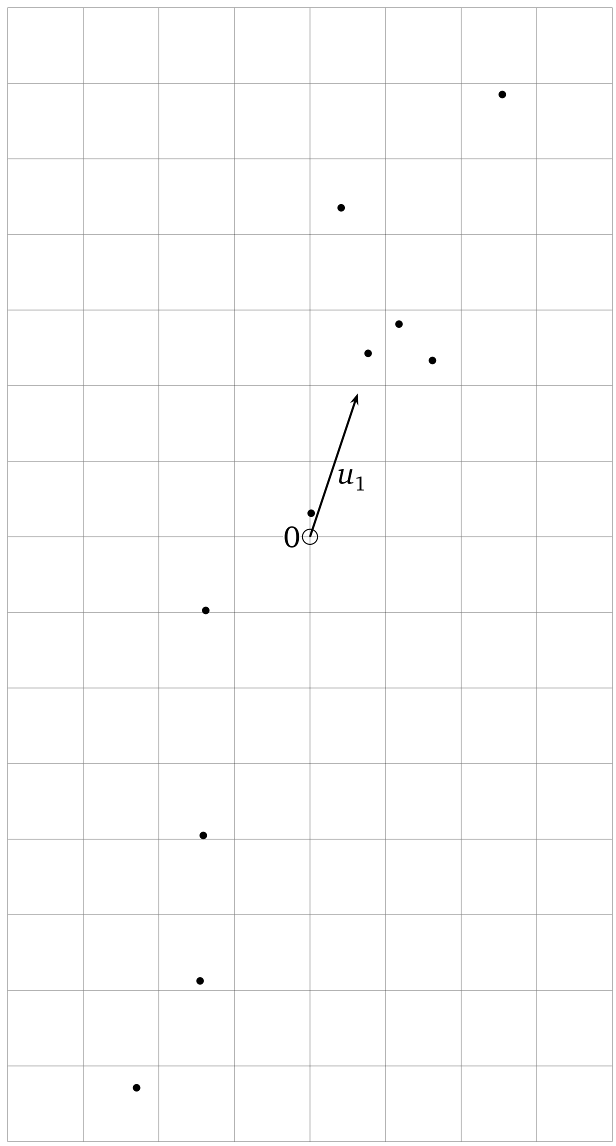
$$A = 7u_1 v_1^T + 0.9u_2 v_2^T,$$

where u_1, u_2 and v_1, v_2 are the singular vectors. The columns of A and the first left singular vector u_1 are drawn below. *Draw and label*:

a) the span of u_1;

b) the columns of $7u_1 v_1^T$ (drawn as dots);

c) the columns of $0.9u_2 v_2^T$ (drawn as arrows).



grid lines are 0.5 units apart

9. Consider the following matrix holding 5 samples of 2 measurements each:

$$A_0 = \begin{pmatrix} 22 & -12 & 24 & -29 & 20 \\ 1 & -11 & 37 & -17 & -35 \end{pmatrix}.$$

a) Subtract the means of the rows of A_0 to obtain the centered matrix A.

b) Compute the covariance matrix $S = \frac{1}{5-1}AA^T$. What is the total variance? What is the covariance of the first row with the second?

c) Compute the variance $s(u)^2$ of your data points in the directions

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

d) Find the eigenvalues λ_1, λ_2 and unit eigenvectors u_1, u_2 of S. What direction is the first principal component? What is the variance of A in that direction? (It should be larger than the variances you computed in **c)**.)

e) Find the orthogonal projections of the columns of A onto the first principal component by computing the first summand $\sigma_1 u_1 v_1^T$ of the SVD of A. (Don't forget to rescale by $\sqrt{5-1}$.)

f) Draw the columns of A, the first principal component you found in **d)**, and the orthogonal projections you found in **e)** on a grid.

10. Let A be a matrix with singular value decomposition

$$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T.$$

Show that A is a centered data matrix (rows sum to zero) if and only if the entries of each right singular vector v_i sum to zero.
[**Hint:** Multiply by the ones vector $\mathbf{1} = (1, 1, \ldots, 1)$.]

11. Let A be a matrix with singular value decomposition

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

Recall that the maximum value of $\|Ax\|$ subject to $\|x\| = 1$ is σ_1, and is achieved at $x = v_1$.

a) Show that the maximum value of $\|Ax\|$ subject to the conditions $\|x\| = 1$ and $x \cdot v_1 = 0$ is equal to σ_2, and is achieved at $x = v_2$.
[**Hint:** If $x \cdot v_1 = 0$ then $Ax = A'x$ for $A' = \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T + \cdots + \sigma_r u_r v_r^T$.]

b) More generally, show that the maximum value of $\|Ax\|$ subject to the conditions $\|x\| = 1$ and $x \cdot v_1 = 0,\ x \cdot v_2 = 0,\ \ldots,\ x \cdot v_j = 0$ is equal to σ_{j+1}, and is achieved at $x = v_{j+1}$.

c) If A has full column rank, show that the *minimum* value of $\|Ax\|$ subject to $\|x\| = 1$ is equal to σ_r, and is achieved at $x = v_r$.

In the language of principal component analysis, this says that v_2 is the direction of *second-largest variance*, etc.