

Math 218D-1: Homework #3

due Wednesday, January 31, at 11:59pm

1. Consider a system of 3 equations in 4 variables. Write the elementary matrices that accomplish the following row operations:

$$\begin{array}{lll} \text{a) } R_2 += 2R_1 & \text{b) } R_1 -= \frac{1}{2}R_3 & \text{c) } R_3 \times = 2 \\ \text{d) } R_3 \div = 2 & \text{e) } R_1 \longleftrightarrow R_3 & \text{f) } R_1 \longleftrightarrow R_2 \end{array}$$

2. Consider a system of 3 equations in 4 variables. Write the row operations that the following elementary matrices perform on that system:

$$\begin{array}{lll} \text{a) } \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{b) } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{c) } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \\ \text{d) } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \text{e) } \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \end{array}$$

3. For each elementary matrix in Problem 2, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row op}} R_2 += R_1 \xrightarrow{\text{undo}} R_2 -= R_1 \xrightarrow{\text{matrix}} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

- a) Explain how to reduce A to a matrix U in REF using three row replacements.
- b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i :

$$U = \underline{\hspace{2cm}} A.$$

- c) Fill in the blank with a product involving the E_i^{-1} :

$$A = \underline{\hspace{2cm}} U$$

- d) Evaluate that product to produce a lower-triangular matrix L with ones on the diagonal such that $A = LU$.

When multiplying elementary matrices, just use row operations!

- e) Explain how to reduce U to the 3×3 identity matrix using three more row operations E_4, E_5, E_6 .

- f) Fill in the blank with a product involving the E_i :

$$A^{-1} = \underline{\hspace{2cm}}.$$

5. Solve the following matrix equations by forward- and back-substitution, using the provided LU decomposition. Check your answers by evaluating Ax .

$$\text{a) } \begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\text{b) } \begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{c) } \begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

6. Compute the $A = LU$ decomposition of the following matrices using the 2-column method. Check your answers by multiplying LU .

$$\text{a) } \begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix} \quad \text{c) } \begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$$

7. Solve the following matrix equations by forward- and back-substitution, using the provided $PA = LU$ decomposition. Check your answers by evaluating Ax .

a)
$$\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix} \begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b)
$$\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} x = \begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

8. Compute a $PA = LU$ decomposition for each of the following matrices, using the 3-column method and performing *maximal partial pivoting*. Check your answers by multiplying PA and LU .

a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$

9. Recall that a *permutation matrix* is a product of elementary matrices for row swaps.
- a) If P is the $n \times n$ elementary matrix for a row swap, explain why $P^{-1} = P = P^T$.
- b) If P is any permutation matrix, show that $P^{-1} = P^T$. [Hint: write $P = P_1 P_2 \cdots P_r$ for elementary row swaps P_i .] Is $P = P^T$ for a general permutation matrix?

10. Express each system of linear equations as a vector equation. For example,

$$\begin{array}{r} x_1 + 2x_2 = 3 \\ -x_1 - x_2 = 4 \end{array} \rightsquigarrow x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

$$\text{a) } \begin{cases} 3x_1 + 2x_2 + 4x_3 = 9 \\ -x_1 + 4x_3 = 2 \end{cases} \quad \text{b) } \begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\text{c) } \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array} \right)$$

11. For each matrix A and vector b , decide if the system $Ax = b$ is consistent. If so, find the parametric form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow x_1 = x_2 + 1.$$

Also answer the following questions: Which variables are free? How many solutions does the system have?

$$\text{a) } A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{b) } A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

$$\text{c) } A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 48 \end{pmatrix}$$

$$\text{d) } A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

$$\text{e) } A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

12. For each matrix A and vector b in Problem 11, find the parametric vector form of the general solution of $Ax = b$ (if the system is consistent). For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the dimension of the solution set?

13. a) Is $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$?
If so, what are the weights?

b) Find a vector that is *not* a linear combination of the columns of the matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}.$$

[**Hint:** for both parts, compare Problem 11. This does not require any additional computation.]

14. The equation $x + 2y = z$ determines a plane in \mathbf{R}^3 . (This is an *implicit equation* for the plane).
- a) What is the coefficient matrix A for this system?
- b) Which are the free variables?
- c) Write the parametric form of the solutions of $x + 2y = z$. This expresses the points on the plane in terms of two *parameters*.
- d) Do the same for the plane defined by $2y = z$. What is different?

15. The equations

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \end{aligned}$$

determine a line \mathbf{R}^3 . (These are *implicit equations* for the line). Write the line in parameterized form: that is, find three linear functions $f_1(t), f_2(t), f_3(t)$ in one variable such that all points on the line have the form $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ for a unique value of t . (Use the free variable as the parameter t .)

16. Let A be a 4×5 matrix with four pivots. Suppose that

$$(\text{column } 1) + 2(\text{column } 3) - (\text{column } 4) = 0.$$

- a) Find a nonzero solution of $Ax = 0$.
- b) Which is the free variable?

[**Hint:** the solutions of $Ax = 0$ are unchanged by Gauss–Jordan elimination.]

17. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix} 1+t \\ 2-t \\ t \end{pmatrix} \quad t \in \mathbf{R}.$$

- 18.** Suppose that A is a 3×3 matrix and b is a vector such that the solution set of $Ax = b$ is a line in \mathbf{R}^3 . How many pivots does A have?
- 19.** Let R be a row echelon form of a matrix A . Explain why the following quantities are all equal to the rank of A :
- a) The number of pivots of A .
 - b) The number of nonzero rows of R .
 - c) The number of columns of A minus the number of free variables.
- 20.** Decide if each statement is true or false, and explain why.
- a) A square matrix has no free variables.
 - b) An invertible matrix has no free variables.
 - c) An $m \times n$ matrix has at most m pivots.
 - d) A wide matrix (more columns than rows) must have a free variable.
 - e) If A is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.