Welcome to Math 218D-1!

Introduction to Linear Algebra

What is Linear Algebra?

The study of (systems of) linear equations

Like: $y=3x+2 \rightsquigarrow -3x+y=2$

(usually put variables on the left & constants on the right)

Or: $\begin{cases} x+y+z=1 \\ \quad\; y-z=-3 \end{cases}$; solve both equations at once

(arrange in columns to keep things tidy)

Linear means: equations that involve only sums of (number)·(variable) or (number)

Not: $xy+z=1$ $\quad x+3=y^2$ $\quad e^x=\cos(y)$

product of variables

power of a variable

complicated functions

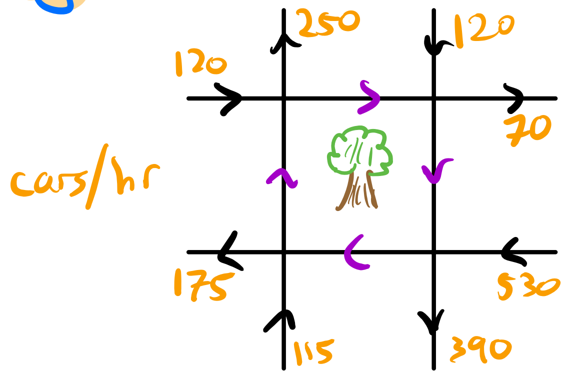
Why linear algebra?

- It's simple enough to understand very well & program computers to do quickly.

- It's powerful enough to solve a huge range of different problems.

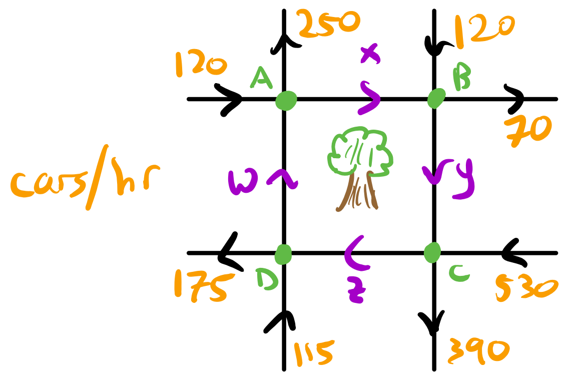
"example"

Eg: Here's a map of roads in the town square:



Question: How many cars/hr travel on the unlabeled roads?

Step 0: When you have an unknown quantity, give it a name!



Observation:
cars entering each intersection = # cars leaving it

A: $120 + w = 250 + x$

B: $120 + x = 70 + y$

C: $530 + y = 390 + z$

D: $115 + z = 175 + w$

(notice nice columns)

$$\leadsto \begin{cases} -x & & & +w & = 130 \\ x & -y & & & = -50 \\ & y & -z & & = -140 \\ & & z & -w & = 60 \end{cases}$$

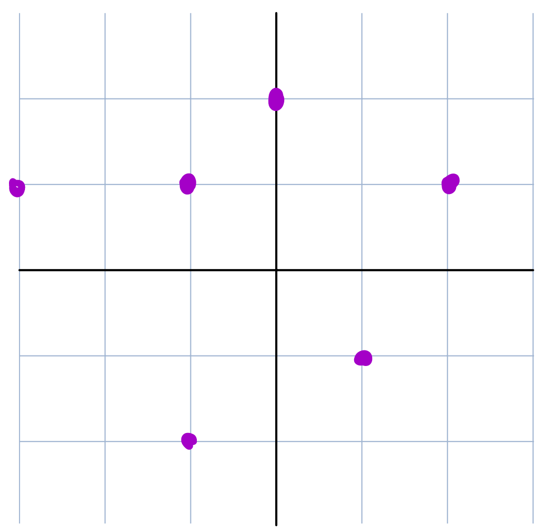
This is a system of 4 linear equations in 4 unknowns!

Question: You know a priori that there are infinitely many solutions. How?

Question: What must be true about the known quantities for a solution to exist?

Linear algebra is a set of tools for solving equations. It is your job to turn your question into a linear algebra problem (that a computer can solve) and interpret the answer.

Eg: An asteroid has been observed at coordinates:
$(0,2)$, $(2,1)$, $(1,-1)$, $(-1,-2)$, $(-3,1)$, $(-1,1)$



Question: What is the most likely orbit? Will the asteroid crash into the Earth?

Fact: The orbit is an ellipse.

Equation for an ellipse:

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

Wait! Isn't this a nonlinear equation? ...

For our points to lie on the ellipse, substitute the coordinates into (x,y) ⇝ these should hold:

$$
\begin{array}{ll}
\overset{x\ \ y}{(0,2)}: & 0 + 4B + 0 + 0 + 2E + F = 0 \\
(2,1): & 4 + B + 2C + 2D + E + F = 0 \\
(1,-1): & 1 + B - C + D - E + F = 0 \\
(-1,-2): & 1 + 4B + 2C - D - 2E + F = 0 \\
(-3,1): & 9 + B - 3C + D - 3E + F = 0 \\
(-1,1): & 1 + B - C - D + E + F = 0
\end{array}
$$

This is a system of six linear equations in 5 variables.

"Note"

NB: The variables are the coefficients B, C, D, E, F. Remember, we're finding the equation of the ellipse.

NB: There is no solution — the points do not lie on an ellipse (perhaps due to measurement error).

Question: What is the best approximate solution?

⇝ "least squares" (week 8)

Answer: [demo]

Historical note: Gauss invented much of what you'll learn to (correctly) predict the orbit of the asteroid Ceres in 1801.

Note on demos: I created these to help give you a geometric understanding of linear algebra.

→ It took a lot of work.

→ Really, it was hard.

→ Why would I do that? I want you to have a geometric understanding.

Upshot: Play with the demos! Don't turn off your brain when we do geometry! You will be expected to draw pictures on exams!

Eg: In a population of rabbits,

(1) Half survive their first year

(2) Half of those survive their second year.

(3) The maximum life span is 3 years.

(4) Each rabbit produces (on average) 0, 6, 8 offspring in years 0, 1, 2, respectively.

Question: How many rabbits will there be in 100 years?

Step 0: Give names to the unknowns.

x_n: # rabbits aged 0 in year n
y_n: # rabbits aged 1 in year n
z_n: # rabbits aged 2 in year n

Rules:
$$\begin{aligned} x_{2021} &= \phantom{\tfrac{1}{2}x_{2020}} \quad 6y_{2020} + 8z_{2020} \\ y_{2021} &= \tfrac{1}{2}x_{2020} \\ z_{2021} &= \phantom{\tfrac{1}{2}x_{2020}} \quad \tfrac{1}{2}y_{2020} \end{aligned}$$

A system of equations of this form is called a difference equation. We'll solve them using eigenvalues & diagonalization (week 10).

[demo] It looks like eventually,

- The population doubles each year
- The ratio of rabbits aged 0 : 1 : 2 is $\approx 16 : 4 : 1$

Comes from: $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{bmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$ w/ eigenvalue 2.

Other examples:

- Google PageRank lets you search the Web with a Markov chain — a special type of difference equation.

- Netflix knows what movies you'll like using the Singular Value Decomposition (weeks 14–15).

Geometry of Solutions

Convention: given a system of linear equations, put the constant term on the right of the =, and put the variables on the left, organized in columns.

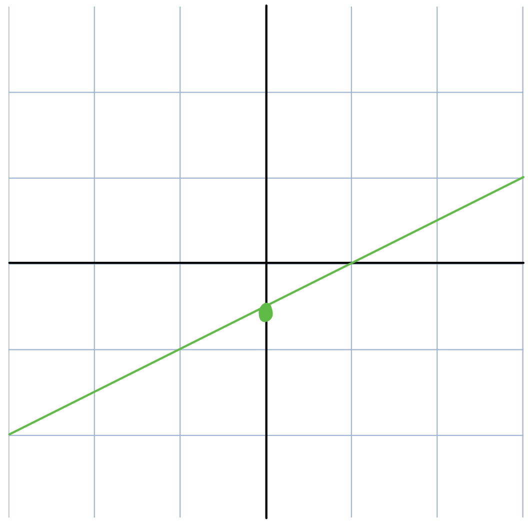
$$\begin{aligned} 120 + w &= 250 + x \\ 120 + x &= 70 + y \\ 530 + y &= 390 + z \\ 115 + z &= 175 + w \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} -x \phantom{{}-y-z} + w &= 130 \\ x - y \phantom{{}-z-w} &= -50 \\ y - z \phantom{{}-w} &= -140 \\ z - w &= 60 \end{aligned}$$

Def: The solution set of a system of equations is the set of all values for the variables making all equations true simultaneously.

Question: What does the solution set of a system of linear equations look like?

One equation in 2 variables:

$x - 2y = 1 \rightsquigarrow y = \frac{1}{2}x - \frac{1}{2}$



One equation in 3 variables:

$x + y + z = 1 \rightsquigarrow z = 1 - x - y$

plane in xyz-space

[demo]

One equation in 4 variables: "3-plane in 4-space"

Note on dimensions: Students often want to say "the fourth dimension is time". Einstein used $\mathbb{R}^4$ (4-space) to model spacetime, but it models lots of other things too. (like traffic around the town square...)

2 equations in 2 variables:

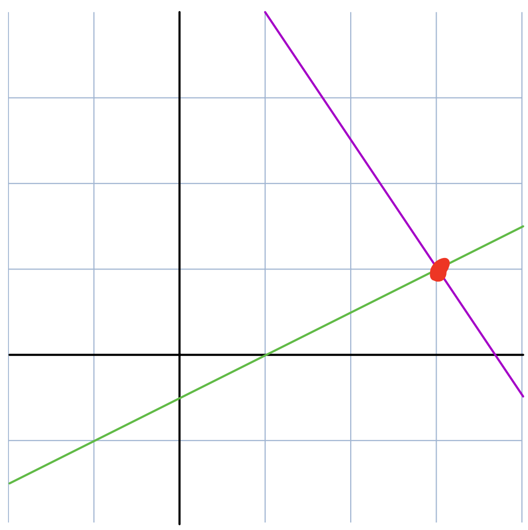
$x - 2y = 1$

$3x + 2y = 11$

Where are both true?

Intersection of 2 lines.

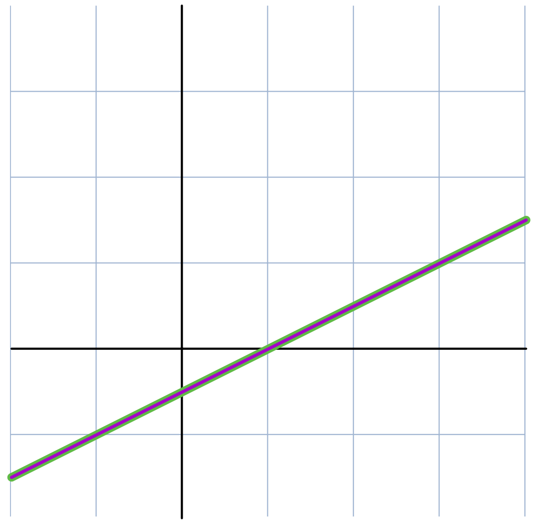
(answer: $(3,1)$)



What else can happen?

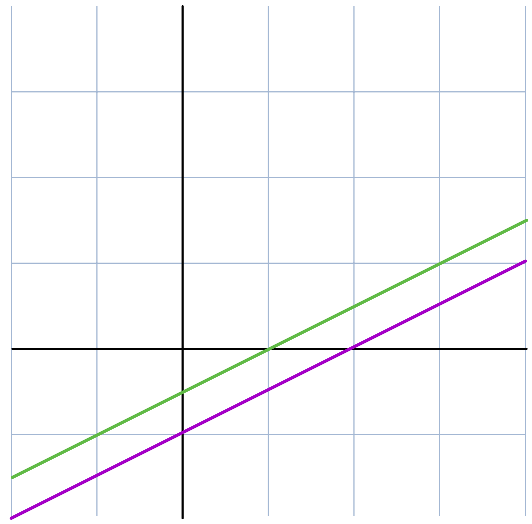
$x - 2y = 1$
$3x - 6y = 3$

Same line: ∞ solutions.



$x - 2y = 1$
$3x - 6y = 6$

Parallel lines: 0 solutions



2 equations in 3 variables:

$$\begin{aligned} x + y + z &= 1 \\ x \quad - z &= 0 \end{aligned}$$

$\rightsquigarrow$ intersection of two planes in space [demo]

In this case, it's a line.

3 equations in 3 variables:

$$\begin{aligned} x + y + z &= 1 \\ x \quad - z &= 0 \\ y \quad &= 0 \end{aligned} \rightsquigarrow \begin{aligned} x &= 1/2 \\ y &= 0 \\ z &= 1/2 \end{aligned}$$

intersection of three planes in space: in this case it's a point.

Question: How many "ways" can 3 planes in space intersect?

Answer: 8

Syllabus Stuff: see the syllabus for details.

- Course materials, calendar, resources, links, etc. are on the course webpage:

https://services.math.duke.edu/~jdr/2324s-218/

- We will use Sakai for:
  - → Announcements
  - → Gradebook
  - → Gradescope

    !! Please use the Gradescope tab on Sakai instead of going to gradescope.com.

  - → Ed Discussion: for asking questions (replaces Piazza).

    !! Don't email us w/math questions! Post it here instead – then everyone sees it & anyone can answer.

  - → WarpWire (see below)

Textbook:

- Strang, "Introduction to Linear Algebra," 5^{th} ed.

We'll only follow this loosely. Also see

- Margalit-Rabinoff, "Interactive Linear Algebra" (on the course website). You'll get a link to a beta version aimed just at this course!

Quizzes: a 10-minute small-group quiz will be held at the beginning of each discussion section. It's very basic — just tests if you've looked over your notes.

Homework: due Wednesday 11:59pm every week.

- Meant to be long and hard: you need practice to learn math, and practice takes time.
- Scan & submit on Gradescope. Use a scanning app!
- Tag the pages on Gradescope with the problems on that page!

Midterms: 2 of them, during discussion slots.

Final: as scheduled by the registrar.

Help! • Come to office hours!

- Ask on Ed Discussion
- See course webpage.

Recorded Lecture:

Basics of vector & matrix algebra.

Watch before Tuesday (on Warp Wire)

HW#1 covers that material.