

The Basis Theorem

Recall from last time:

Basis of $\mathbb{R}^n \equiv$ cols of an invertible $n \times n$ matrix

For an $n \times n$ matrix,

full col rank $\Leftrightarrow$ invertible $\Leftrightarrow$ full row rank

In terms of columns, n vectors in $\mathbb{R}^n$

spans $\mathbb{R}^n \Leftrightarrow$ linearly independent

This is a special case of the basis theorem.

Basis Theorem: Let V be a subspace of dim d

(1) If d vectors span V then they're a basis

(2) If d vectors in V are LI then they're a basis.

So if you have the correct number of vectors, you only need to check one of spans / LI.

Eg: • Two noncollinear vectors in a plane form a basis.

• Two vectors that span a plane form a basis.

This is how the Basis Thm makes our intuition precise.

Geometry of Dot Products

We are now aiming to find the "best" approximate solution of $Ax=b$ when no actual solution exists.

Eg: find the best-fit ellipse through these points from the 12th lecture...

Q: How close can Ax get to b ?

$$\text{Col}(A) = \{Ax : x \in \mathbb{R}^n\}$$

so this means: what is the closest vector $\hat{b}$ in $\text{Col}(A)$ to b ?

A: $b - \hat{b}$ is perpendicular to $\text{Col}(A)$

[demo]

So we want to understand what vectors are perpendicular to a subspace.

We will study the geometric notion of "perpendicular" using the algebra of dot products.

Recall: $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \rightarrow v \cdot w = x_1 y_1 + \dots + x_n y_n = v^T w$

$(v^T w = (x_1 \dots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (x_1 y_1 + \dots + x_n y_n) = (v \cdot w))$

1x1 matrix

Dot products measure length & angles (eg. 90°)

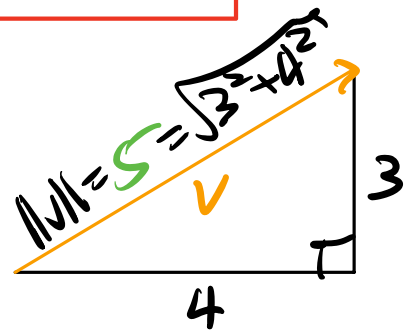
→ geometric questions about length & angles become algebraic questions about dot products.

Recall: If $v = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ then
 $v \cdot v = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$

Def: The length of v is

$$\|v\| = \sqrt{v \cdot v} \quad \text{ie} \quad \|v\|^2 = v \cdot v$$

This makes sense by the
Pythagorean theorem: $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$



Sanity Check: $c \in \mathbb{R} \quad v \in \mathbb{R}^n$

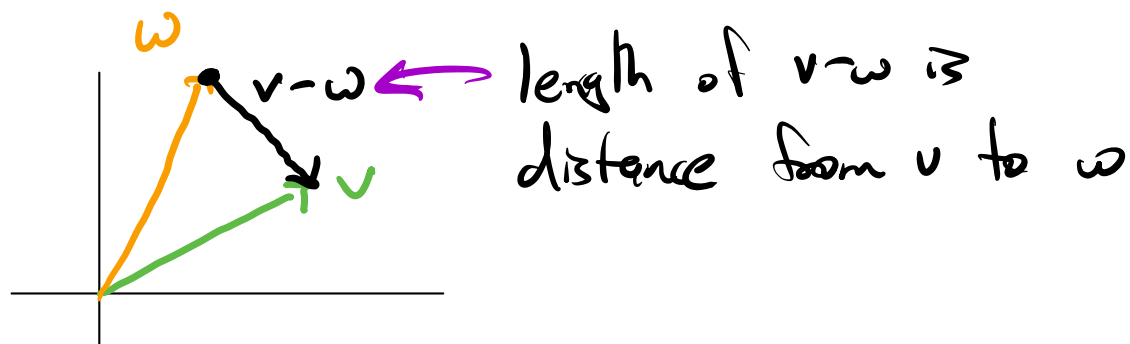
$$\begin{aligned} \|cv\| &= \left\| c \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \right\| = \sqrt{(cx_1)^2 + \dots + (cx_n)^2} \\ &= |c| \cdot \sqrt{x_1^2 + \dots + x_n^2} = |c| \cdot \|v\| \quad \checkmark \end{aligned}$$

$$\|cv\| = |c| \cdot \|v\|$$

Eg: $2v$ is twice as long as v .

So is $-2v$.

Def: The distance from v to w is $\|v-w\| = \|w-v\|$

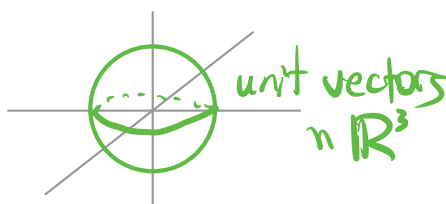
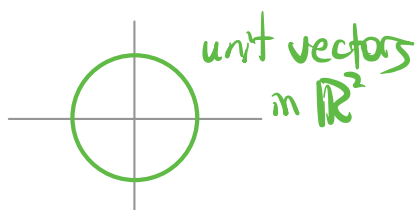


Def: A unit vector is a vector of length 1.
ie $\|v\| = 1$ ie. $\|v\|^2 = v \cdot v = 1$

If $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then v is a unit vector

$$\iff x_1^2 + \dots + x_n^2 = 1$$

$\iff v$ lies on the unit $(n-1)$ -sphere
($n=2$: unit circle)



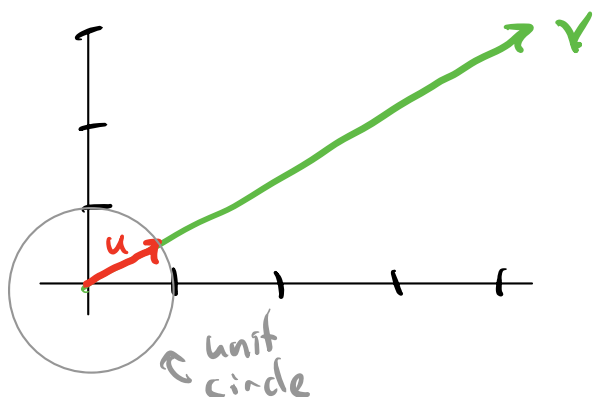
If $v \neq 0$, the unit vector in the direction of v
is the vector

$$u = \frac{1}{\|v\|} \cdot v = \frac{v}{\|v\|} \quad (\text{scalar} \times \text{vector})$$

NB: $\|u\| = \left| \frac{1}{\|v\|} \right| \cdot \|v\| = \frac{\|v\|}{\|v\|} = 1$ ✓

Eg: $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\|v\| = \sqrt{3^2 + 4^2} = 5$

$u = \frac{1}{\|v\|} v = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$

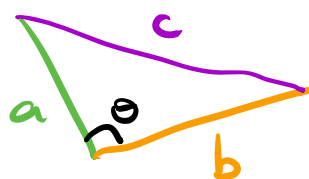


NB: all unit vectors in $\mathbb{R}^2$ are on the unit circle.

What about $v \cdot w$ for $v \neq w$?

Law of Cosines:

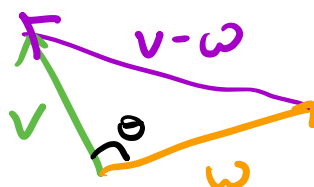
$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



Vector Version:

$$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos \theta$$

($a=\|v\|$ $b=\|w\|$ $c=\|v-w\|$)



Algebra: "left hand side" LHS: $\|v-w\|^2 := (v-w) \cdot (v-w)$

FOIL $= v \cdot v + w \cdot w - 2v \cdot w$

$$= \|v\|^2 + \|w\|^2 - 2v \cdot w$$

"right hand side" RHS $= \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos \theta$

canceling $\boxed{v \cdot w = \|v\|\|w\|\cos \theta \text{ or } \cos \theta = \frac{v \cdot w}{\|v\|\|w\|} \text{ (if } v, w \neq 0)}$

Def: The **angle** from v to w ($v, w \neq 0$) is

$$\theta := \cos^{-1} \left(\frac{v \cdot w}{\|v\| \|w\|} \right)$$

NB: $|\cos \theta| = \left| \frac{v \cdot w}{\|v\| \|w\|} \right| \in [0, 1]$

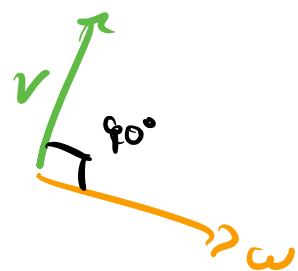
$$\Rightarrow |v \cdot w| \leq \|v\| \cdot \|w\|$$

Schwartz Inequality: $|v \cdot w| \leq \|v\| \cdot \|w\|$ ✓

Def: Vectors v and w are **orthogonal** or **perpendicular**, written $v \perp w$, if $v \cdot w = 0$

This says that either:

- $v = 0$ or $w = 0$ (or both), or
- $\cos(\theta) = 0 \iff \theta = \pm 90^\circ$



NB: The zero vector is orthogonal to **every vector**:
 $0 \cdot v = 0$ for all v

Orthogonality

We want to know: "which vectors are $\perp$ a subspace?"
Let's start with: "which vectors are $\perp$ some vector?"

Eg: Find all vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

We need to solve $v \cdot x = 0$

$$\Leftrightarrow v^T x = 0$$

This is just $\text{Nul}(v^T)$:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \rightsquigarrow x_1 + x_2 + x_3 = 0$$

$$\begin{array}{l} \text{PF} \rightsquigarrow \\ x_1 = -x_2 - x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{array}$$

$$\text{PVP} \rightsquigarrow x = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{[demo]} \rightsquigarrow \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ a plane}$$

$$\text{Check: } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \checkmark$$

Eg: Find all vectors orthogonal to $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

We need to solve $\begin{cases} v_1^T \cdot x = 0 \\ v_2^T \cdot x = 0 \end{cases} \rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \end{cases}$

Equivalently, $\begin{pmatrix} -v_1^T \\ -v_2^T \end{pmatrix} \cdot x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \end{pmatrix} = 0$

So we want $\text{Nul} \begin{pmatrix} -v_1^T \\ -v_2^T \end{pmatrix} = \text{Nul} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{PF}} \begin{cases} x_1 = -x_2 \\ x_2 = x_2 \\ x_3 = 0 \end{cases}$$

$$\xrightarrow{\text{PVF}} x = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\leadsto \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{a line}$$

$$\text{Check: } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \checkmark$$

[demo]

NB: If $x \perp v_1$ and $x \perp v_2$ then

$$x \cdot (av_1 + bv_2) = a x \cdot v_1 + b x \cdot v_2 = a \cdot 0 + b \cdot 0 = 0$$

So x is orthogonal to every vector in $\text{Span}\{v_1, v_2\}$

[demo again]

More generally,

$$\left\{ v \in \mathbb{R}^n : \begin{array}{l} v \text{ is orthogonal} \\ \text{to every vector} \\ \text{in } \text{Span}\{v_1, \dots, v_n\} \end{array} \right\} = \text{Nul} \begin{pmatrix} -v_1^T - \\ \vdots \\ -v_n^T - \end{pmatrix}$$

↗ This is awkward to say - let's give it a name.

Def: Let V be a subspace of $\mathbb{R}^n$.

The orthogonal complement of V is

$$V^\perp = \left\{ w \in \mathbb{R}^n : \begin{array}{l} w \text{ is orthogonal to} \\ \text{every vector in } V \end{array} \right\}$$

NB: Note the difference in notations =

- $V^\perp$ is the orthogonal complement of a subspace
- A^T is the transpose of a matrix.

NB: If x is in both V and $V^\perp$ then x is orthogonal to itself:

$$x \cdot x = 0 \Rightarrow x = 0, \text{ so } \underset{\text{intersect}}{V \cap V^\perp} = \{0\}$$

So we showed above:

$$\text{Span}\{v_1, \dots, v_n\}^\perp = \text{Nul} \begin{pmatrix} -v_1^T \\ \vdots \\ -v_n^T \end{pmatrix}$$

Eg: $V = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \Rightarrow V^\perp = \text{Nul}\begin{pmatrix} 1 & -1 \end{pmatrix}$

Eg: $V = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \Rightarrow V^\perp = \text{Nul}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Eg: $\{0\}^\perp = \mathbb{R}^n \quad (\mathbb{R}^n)^\perp = \{0\}$

Fact: $V^\perp$ is also a subspace of $\mathbb{R}^n$.

Check:

(1) Let $x, y \in V^\perp$. So $x \cdot v = 0$ and $y \cdot v = 0$ for every $v \in V$. So $(x+y) \cdot v = x \cdot v + y \cdot v = 0 + 0$ for every $v \in V \Rightarrow x+y \in V^\perp$.

(2) Let $x \in V^\perp$, $c \in \mathbb{R}$. So $x \cdot v = 0$ for every $v \in V$.
So $(cx) \cdot v = c(x \cdot v) = c \cdot 0 = 0$ for every $v \in V$
 $\leadsto cx \in V^\perp$.

(3) $0 \cdot v = 0$ for every $v \in V \leadsto 0 \in V^\perp$.

Or:

Every subspace is a span, and the orthogonal complement of a span is a null space (which is a subspace).

Facts: Let V be a subspace of $\mathbb{R}^n$.

(1) $\dim(V) + \dim(V^\perp) = n$ [demos]

(2) $(V^\perp)^\perp = V$

NB: (2) says V and $V^\perp$ are orthogonal complements of each other. Subspaces come in orthogonal complement pairs.

Orthogonality of the Four Subspaces

Recall: If someone gives you a subspace,
Step 0 is to write it as a column space or a null space. So we want to understand $\text{Col}(A)^\perp$ & $\text{Nul}(A)^\perp$.

Let $A = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$. Then

$$\text{Col}(A)^\perp = \text{Span}\{v_1, \dots, v_n\}^\perp = \text{Nul} \begin{pmatrix} -v_1^T \\ \vdots \\ -v_n^T \end{pmatrix} = \text{Nul}(A^T)$$

$$\text{Col}(A)^\perp = \text{Nul}(A^T)$$

Take $(\cdot)^\perp$ $\text{Col}(A) = (\text{Col}(A)^\perp)^\perp = \text{Nul}(A^T)^\perp$

replace A by A^T $\text{Row}(A) = \text{Col}(A^T) = \text{Nul}(A)^\perp$

and $\text{Row}(A)^\perp = \text{Nul}(A)$

Orthogonality of the Four Subspaces:

$$\text{Col}(A)^\perp = \text{Nul}(A^T)$$

$$\text{Nul}(A)^\perp = \text{Row}(A)$$

$$\text{Nul}(A^T)^\perp = \text{Col}(A)$$

$$\text{Row}(A)^\perp = \text{Nul}(A)$$

This says the two **row picture** subspaces $\text{Row}(A)$, $\text{Nul}(A)$ are orthogonal complements, & the two **column picture** subspaces $\text{Col}(A)$, $\text{Nul}(A^T)$ are orthogonal complements.

Eg: $V = \{x \in \mathbb{R}^3 : \begin{matrix} x+2y=z \\ x+y+z=0 \end{matrix}\}$. Find a basis for $V^\perp$.

Step 0: $V = \text{Nul} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow V^\perp = \text{Row} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$V^\perp = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$: **no elimination needed!**

Eg: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

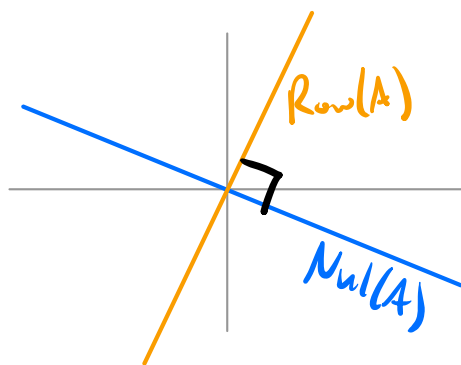
$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array} \right)$$

$$\rightsquigarrow \text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \quad \text{Nul}(A^T) = \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$\text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{Row}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Row Picture



Column Picture

