

# Determinants

## What we've done:

- Solve  $Ax=b$   
(Gauss-Jordan, LU, PVE, ...)
- Approximately solve  $Ax=b$   
(orthogonality, projections, QR, ...)

## What's next:

- Solve  $Ax=\lambda x$

This is the **eigenvalue problem** used in difference equations (rabbit population) & ODEs. It deals exclusively with **square matrices**.

The **determinant** of a **square** matrix is a number that satisfies many **magical properties**. I'll define it by telling you how to compute it using **row operations**.

→ Next time: other ways to compute it.

Def: The determinant of a square matrix  $A$  is a number  $\det(A)$  or  $|A|$  satisfying:

(1) If  $A \xrightarrow{R_i + cR_j} B$  then  $\det(A) = \det(B)$ .

(2) If  $A \xrightarrow{R_i \times c} B$  then  $\det(A) = \frac{1}{c} \det(B)$ .

(3) If  $A \xrightarrow{R_i \leftrightarrow R_j} B$  then  $\det(A) = -\det(B)$

(4)  $\det(I_n) = 1$ .

Consequence: if  $A$  has a zero row then  $\det(A) = 0$

Eg:  $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(2)} -\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -0 & -0 & -0 \end{pmatrix}$

$$\Rightarrow \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Consequence: if  $A$  is (upper/lower) triangular then  $\det(A) =$  product of diagonal entries

$\det \begin{pmatrix} \text{triangular} \\ \text{matrix} \end{pmatrix} = \text{product of the diagonal entries}$

↘ eg. REF

Eg:  $\det \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\substack{R_1 \times = \frac{1}{a} \quad R_2 \times = \frac{1}{b} \\ R_3 \times = \frac{1}{c}}]{(2)} abc \det \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

$\xrightarrow[\text{replacements}]{\text{row (1)}} abc \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{(4)}{=} abc$

What if  $b=0$  though? (or  $a$ ? or  $c$ ?)

$\det \begin{pmatrix} a & * & * \\ 0 & 0 & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\text{replacements}]{\text{row (1)}} \det \begin{pmatrix} a & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$

$= 0 = a \cdot 0 \cdot c \quad \checkmark$

A REF matrix is triangular, so you can compute  $\det(A)$  by Gaussian elimination!

Eg:  $\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{(3)} -\det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$

$\xrightarrow{(1)} -\det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{(1)} -\det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

$= -2$

NB: You get the same number for  $\det(A)$  no matter which row operations you do!

Eg:  $\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow[(3)]{R_1 \leftrightarrow R_3} \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$$\xrightarrow[(1)]{R_2 - R_1} \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow[(1)]{R_3 - R_2} \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = -2 \checkmark$$

Gaussian elimination is the fastest general algorithm for computing the determinant of a matrix (with known entries).

**Procedure:** To compute  $\det(A)$ , run Gaussian elimination:  $A \xrightarrow[\text{operations}]{\text{row}} U$ . Then

$$\det(A) = (-1)^{\# \text{ row swaps}} \cdot \frac{1}{\prod (\text{row scaling})} \prod (\text{diagonal entries of } U)$$

**NB:** You don't need to do row scaling operations to run Gaussian elimination, so **this term** usually does not appear.

**NB:** Row operations multiply  $\det$  by a **nonzero** scalar:

$$A \xrightarrow[\text{op}]{\text{row}} B \implies \det(B) = (\text{nonzero number}) \cdot \det(A).$$

Eg:  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

• If  $a \neq 0$ :

$$\det(A) \xrightarrow{R_2 \leftarrow \frac{c}{a} R_1} \det \begin{pmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{pmatrix} \\ = a(d - \frac{c}{a}b) = ad - bc$$

• If  $a = 0$ :

$$\det(A) \xrightarrow{R_1 \leftrightarrow R_2} -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \\ = -bc = ad - bc$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

## Magical Properties of the Determinant:

(1) **Existence**: There exists a number  $\det(A)$  satisfying defining properties (1) - (4).

(2) **Invertibility**:  $A$  is invertible  $\Leftrightarrow \det(A) \neq 0$

(3) **Multiplicativity**:  $\det(AB) = \det(A)\det(B)$   
and  $\det(A) \neq 0 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

(4) **Transposes**:  $\det(A^T) = \det(A)$

We'll only prove (2) in class. See ILA for the rest.

## Existence

(1) says: You get the same number for  $\det(A)$  no matter which row ops you do!

## Invertibility

Proof: If  $U$  is a REF of  $A$  then

$\det(U)$  = product of diagonal entries

$\det(U) \neq 0 \iff$  all diagonal entries are nonzero

$\iff A$  has  $n$  pivots

$\iff A$  is invertible

We know  $\det(A) = (\text{nonzero scalar}) \cdot \det(U)$ ,

so also  $\det(A) \neq 0 \iff \det(U) \neq 0$ . ✓

Eg:  $\det \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} = (-1)(-3) - (1)(3) = 0$

so  $\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$  is **singular** (not invertible)

**NB:** If the columns of  $A$  are **linearly dependent** then  $A$  does not have full column rank  $\Rightarrow$  not invertible  $\Rightarrow \det(A) = 0$ . Likewise for rows (take transposes).

$A$  has **linearly dependent** rows or columns  $\Rightarrow \det(A) = 0$

## Multiplicativity

Eg:  $\det \left[ \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{100} \right]$

$$= \det \left[ \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{99} \right]$$

$$= \det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \det \left[ \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{99} \right]$$

$$= \dots = \left[ \det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \right]^{100} = (-2)^{100}$$

More generally,

$$\det(A^n) = \det(A)^n \quad \text{for all } n \geq 0 \\ \text{(and } n < 0 \text{ if } \det(A) \neq 0)$$

Eg: Say  $A$  has a  $PA = LU$  decomposition.

$$\det(L) = \det \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix} = 1$$

You get  $P$  by doing row swaps on  $I_n$ , so

$$\det(P) = (-1)^{\# \text{row swaps}}$$

Hence

$$(-1)^{\# \text{row swaps}} \det(A) = \det(PA)$$

$$= \det(LU) = \det(L) \det(U)$$

$$= \det(U)$$

This recovers the formula on p.4 (we did no row scaling operations).

# Transposes

The transpose property says that  $\det(A)$  satisfies (1)-(3) for **column operations** too: they're just row operations on  $A^T$ .

Eg:  $\det \begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{\underline{\underline{C_1 - 4C_3}}} \det \begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

$\quad \quad \quad \parallel \text{transpose} \quad \quad \quad \parallel \text{transpose}$

$\det \begin{pmatrix} 2 & 3 & 4 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} \xrightarrow{\underline{\underline{R_1 - 4R_3}}} \det \begin{pmatrix} -14 & -9 & 0 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$

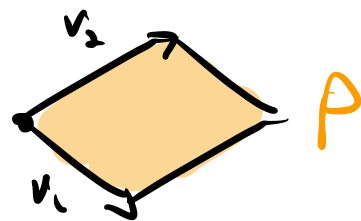
So we can compute  $\det$  using column ops:

$$\det \begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{\underline{\underline{C_1 - 4C_3}}} \det \begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{\underline{\underline{C_1 + 9C_2}}} \det \begin{pmatrix} 49 & 7 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= 49$$

# Determinants and Volumes

Where do properties (1) - (4) come from?

Two vectors  $v_1, v_2 \in \mathbb{R}^2$  determine ("span") a **parallelogram**:



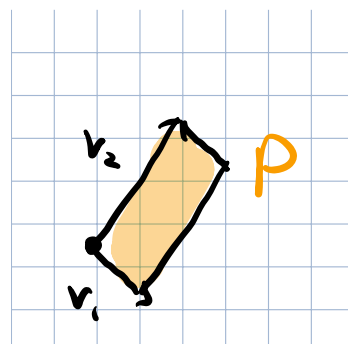
$$P = \{x_1 v_1 + x_2 v_2 : x_1, x_2 \in [0, 1]\}$$

Fact:  $\text{area}(P) = \left| \det \begin{pmatrix} -v_1^T & - \\ -v_2^T & - \end{pmatrix} \right|$

Eg:  $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$   $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

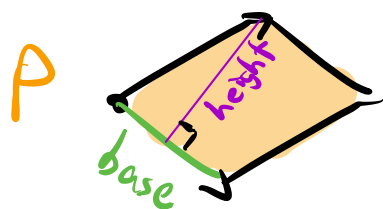
$$\text{area}(P) = \left| \det \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \right|$$

$$= |(3)(1) - (2)(-1)| = 5$$

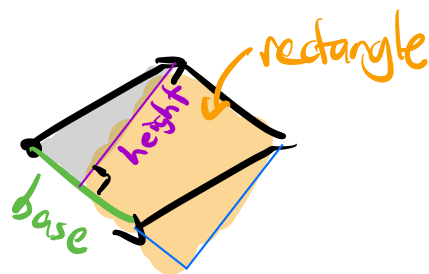


Why? Let's check that  $\text{area}(P)$  satisfies the four defining properties (1) - (4) of the determinant.

NB:  $\text{area}(P) = \text{base} \times \text{height}$ :



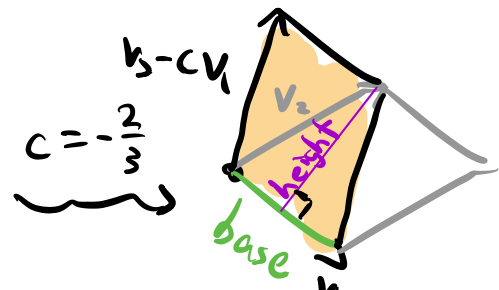
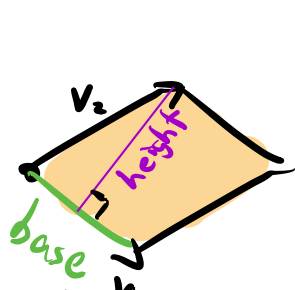
cut &  
rearrange



(1) Row replacement

$$v_2 \rightsquigarrow v_2 + cv_1$$

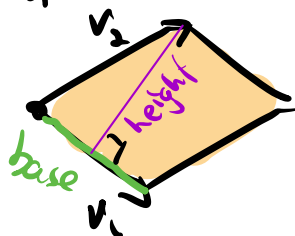
area = base  $\times$  ht : unchanged



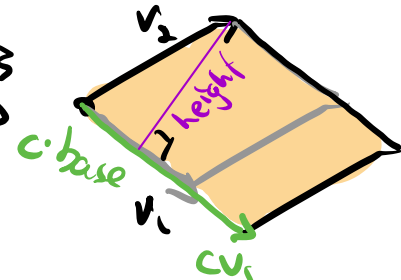
(2) Row scaling

$$v_1 \rightsquigarrow cv_1$$

base scaled by  $|c| \Rightarrow$  base  $\times$  ht : scaled by  $|c|$



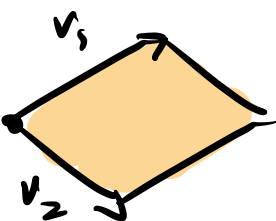
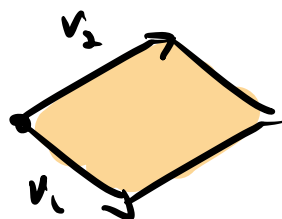
$$c = 1.5$$



(3) Row Swap

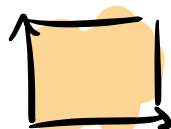
$$v_1 \leftrightarrow v_2$$

area unchanged =  $|\det|$



(4)  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

area = 1.

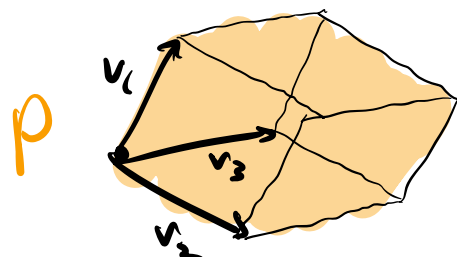


Q: minus sign?  
HW 9

This generalizes as follows (same reasoning):

Def: The **parallelpiped** determined ("spanned") by  $n$  vectors

$$P = \{x_1 v_1 + \dots + x_n v_n : x_1, \dots, x_n \in [0, 1]\}$$



Thm (Determinants & Volumes):

$$\text{volume}(P) = \left| \det \begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix} \right|$$

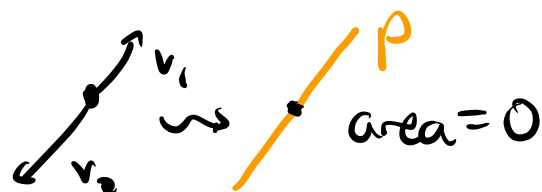
NB: When  $n=1$ , "volume" = "length":  
 $\text{length}(a) = |a|$  

NB: When  $n=2$ , "volume" = "area".

Question: When is  $\text{volume}(P) = 0$ ?

When  $P$  is **squashed flat**:

ie when  $v_1, \dots, v_n$  are



**linearly dependent** ( $\Rightarrow \det(\dots) = 0$ )

NB: In multivariable calc, you approximate shapes by tiny cubes, which turn into tiny parallelepipeds after applying a function. This is why determinants appear in the **change of variables** formula for integrals.

if  $(y_1, \dots, y_n) = f(x_1, \dots, x_n)$  then

$$dy_1 \cdots dy_n = \det(\partial y_i / \partial x_j) dx_1 \cdots dx_n$$

