Linear Independence of Eigenvectors

Recall from last time: to diagonalize an $n \times n$ matrix A:

(1) Compute $p(\lambda) = \det(A - \lambda I_n)$
(2) Solve $p(\lambda) = 0$ to find the eigenvalues
(3) Find a basis for each eigenspace
(4) Combine all these bases.

☆ • If you end up with n vectors, they're LI
• Otherwise A is not diagonalizable

In ☆ we need to justify why the eigenvectors are LI.

Fact: If $w_1, \dots, w_p$ are eigenvectors of A with different eigenvalues then $\{w_1, \dots, w_p\}$ is LI.

Here's how the Fact implies ☆. Suppose

- $\{w_1, w_2\}$ is a basis for the λ_1-eigenspace
- $\{w_3\}$ is a basis for the λ_2-eigenspace.

I claim $\{w_1, w_2, w_3\}$ is LI.

Suppose $x_1 w_1 + x_2 w_2 + x_3 w_3 = 0$. We need $x_1 = x_2 = x_3 = 0$.

- $x_1 w_1 + x_2 w_2$ is in the λ_1-eigenspace
- Since $(x_1 w_1 + x_2 w_2) + x_3 w_3 = 0$, the Fact implies $x_1 w_1 + x_2 w_2 = 0$ and $x_3 w_3 = 0$ (so $x_3 = 0$)
- Since $\{w_1, w_2\}$ is LI, this implies $x_1 = x_2 = 0$ ✓

Proof of the Fact: Say $Aw_i = \lambda_i w_i$ and all of the $\lambda_1, \ldots, \lambda_p$ are distinct. Suppose $\{w_1, \ldots, w_p\}$ is LD. Then for some i, $\{w_1, \ldots, w_i\}$ is LI but $w_{i+1} \in \mathrm{Span}\{w_1, \ldots, w_i\}$, so

$$w_{i+1} = x_1 w_1 + \cdots + x_i w_i$$

$$\Rightarrow Aw_{i+1} = A(x_1 w_1 + \cdots + x_i w_i)$$

$$\Rightarrow \lambda_{i+1} w_{i+1} = \lambda_1 x_1 w_1 + \cdots + \lambda_i x_i w_i$$

If $\lambda_{i+1} = 0$ then $\lambda_1 x_1 w_1 + \cdots + \lambda_i x_i w_i = 0 \xRightarrow[\text{LI}]{\{w_1, \ldots, w_i\}}$ $x_1 = \cdots = x_i = 0$ (because $\lambda_1, \ldots, \lambda_i \neq 0$), so $w_{i+1} = 0$, which can't happen because w_{i+1} is an eigenvector.

If $\lambda_{i+1} \neq 0$ then

$$w_{i+1} = \frac{\lambda_1}{\lambda_{i+1}} x_1 w_1 + \cdots + \frac{\lambda_i}{\lambda_{i+1}} x_i w_i$$

Subtract $w_{i+1} = x_1 w_1 + \cdots + x_i w_i$

$$\rightsquigarrow 0 = \left(\frac{\lambda_1}{\lambda_{i+1}} - 1\right) x_1 w_1 + \cdots + \left(\frac{\lambda_i}{\lambda_{i+1}} - 1\right) x_i w_i$$

But $\lambda_j \neq \lambda_{i+1}$ for $j \leq i$, so $\frac{\lambda_j}{\lambda_{i+1}} - 1 \neq 0$

$$\Rightarrow x_1 = \cdots = x_i = 0$$

which is impossible, as before. ✓

Consequence: If A has n (different) eigenvalues then A is diagonalizable.

Indeed, if $\lambda_1, \dots, \lambda_n$ are eigenvalues and

$$A w_1 = \lambda_1 w_1, \dots, A w_n = \lambda_n w_n$$

then $\{w_1, \dots, w_n\}$ is an eigenbasis by the Fact.

We'll give a more general criterion (AM/GM) next time.

Matrix Form of Diagonalization

Thm: A is diagonalizable $\Longleftrightarrow$ there exists an invertible matrix C and a diagonal matrix D such that

$$A = CDC^{-1}$$

In this case the columns of C form an eigenbasis & the diagonal entries of D are the corresponding eigenvalues.

$$C = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \qquad Aw_i = \lambda_i w_i$$

same order: $w_i \leftrightarrow \lambda_i$

Eg: $A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \Longrightarrow A = CDC^{-1}$ for

$$\begin{array}{cc} & \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \leftarrow (\text{last time}) \\ C = & \begin{pmatrix} 32 & 2 & 18 \\ 4 & -1 & -3 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -3/2 \end{pmatrix}$$

Eg: $A = \begin{pmatrix} 14 & -18 & -33 \\ -12 & 20 & 33 \\ 12 & -18 & -31 \end{pmatrix} \Longrightarrow A = CDC^{-1}$ for

$$\begin{array}{cc} & \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \leftarrow (\text{last time}) \\ C = & \begin{pmatrix} 3 & 11 & 1 \\ 2 & 0 & -1 \\ 0 & 4 & 1 \end{pmatrix} \end{array} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Proof:

$$C\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = x_1w_1 + \cdots + x_nw_n$$

$$\Rightarrow C^{-1}(x_1w_1 + \cdots + x_nw_n) = \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

Any vector has the form $v = x_1w_1 + \cdots + x_nw_n$, and two matrices are equal if they act the same on every vector. So check:

$$\begin{aligned}
CDC^{-1}v &= CDC^{-1}(x_1w_1 + \cdots + x_nw_n)\\
&= C\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\\
&= \begin{pmatrix}| & & |\\ w_1 & \cdots & w_n\\ | & & |\end{pmatrix}\begin{pmatrix}\lambda_1x_1\\ \vdots\\ \lambda_nx_n\end{pmatrix} = x_1\lambda_1w_1 + \cdots + x_n\lambda_nw_n\\
&= A(x_1w_1 + \cdots + x_nw_n) = Av
\end{aligned}$$

✓

NB: If $A = CDC^{-1}$ then

$$\begin{aligned}
A^k = (CDC^{-1})^k &= (CD\cancel{C^{-1}})(\cancel{C}D\cancel{C^{-1}})\cdots(\cancel{C}DC^{-1})\\
&= CD^kC^{-1} = C\begin{pmatrix}\lambda_1^k & & 0\\ & \ddots & \\ 0 & & \lambda_n^k\end{pmatrix}C^{-1}
\end{aligned}$$

This is a closed form expression for A^k in terms of k: much easier to compute!

$$\boxed{A^k = CD^kC^{-1}}$$

← this matrix has n^2 entries that are functions of k

Compare: $A^k(x_1w_1 + \cdots + x_nw_n) = \lambda_1^k x_1 w_1 + \cdots + \lambda_n^k x_n w_n$ (vector form of the same identity).

Eg: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is diagonal:

$$Ae_1 = 2e_1 \quad Ae_2 = 3e_2 \quad Ae_3 = 4e_3$$

So $\{e_1, e_2, e_3\}$ is an eigenbasis $\leadsto$ can take $C = I_3$, so the diagonalization is

$$A = I_3 A I_3$$

Q: What if we take e_2 to be our first eigenvector?

NB: A matrix is diagonal $\iff$ the unit coordinate vectors $e_1, \ldots, e_n$ are eigenvectors.

Geometry of Diagonalizable Matrices

When A is diagonalizable, every vector can be written as a linear combination of eigenvectors, so multiplication by A is reduced to scalar multiplication:

$$A(x_1w_1 + \cdots + x_nw_n) = \lambda_1 x_1 w_1 + \cdots + \lambda_n x_n w_n.$$

What does this mean geometrically?

→ Expanding in an eigenbasis and scalar multiplication can both be formulated geometrically!

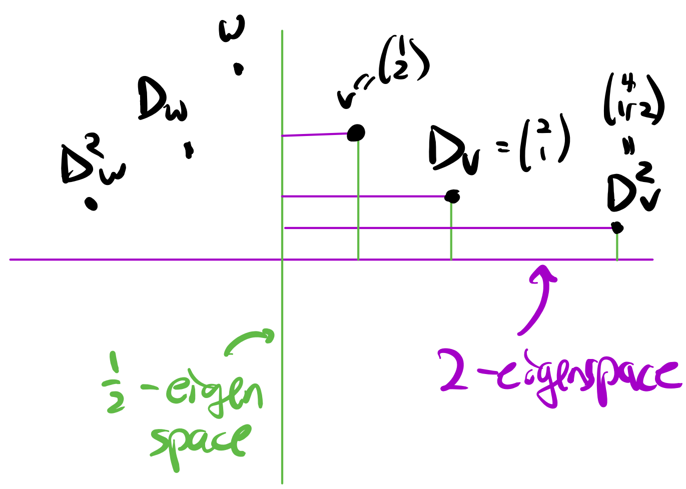
NB: "Visualizing" a matrix means understanding how x relates to Ax: think of A as a function

$$\underset{\text{input}}{x} \rightsquigarrow \underset{\text{output}}{Ax}$$

Eg: $D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ so $D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 1/2\, y \end{pmatrix}$

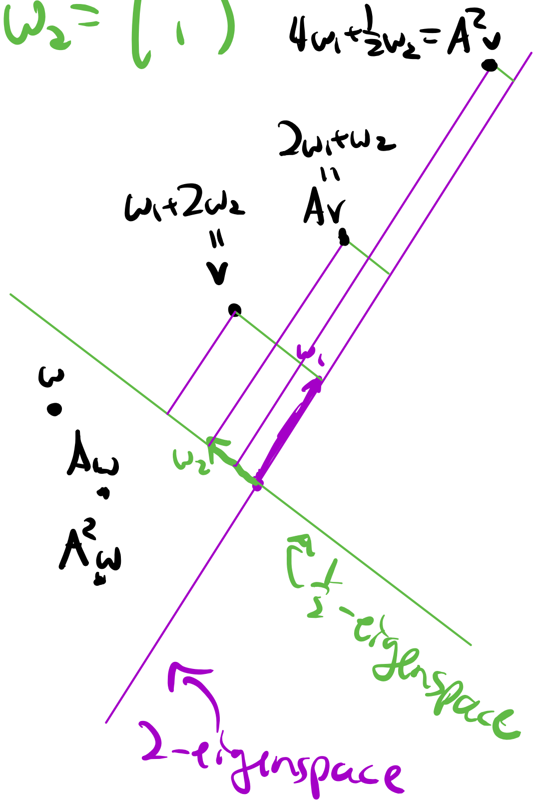
- scales the x-direction by 2
- scales the y-direction by 1/2

[demo]



Eg: $A = \frac{1}{10}\begin{pmatrix} 11 & 6 \\ 9 & 14 \end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - \frac{5}{2}\lambda + 1 = (\lambda - 2)(\lambda - \frac{1}{2})$

$\lambda_1 = 2 \quad w_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \lambda_2 = \frac{1}{2} \quad w_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



Expand in the eigenbasis!
(think in terms of LCs of w_1, w_2)

$$A(x_1 w_1 + x_2 w_2) = 2x_1 w_1 + \frac{1}{2} x_2 w_2$$

- scales the w_1-direction by 2
- scales the w_2-direction by $1/2$

[demo]

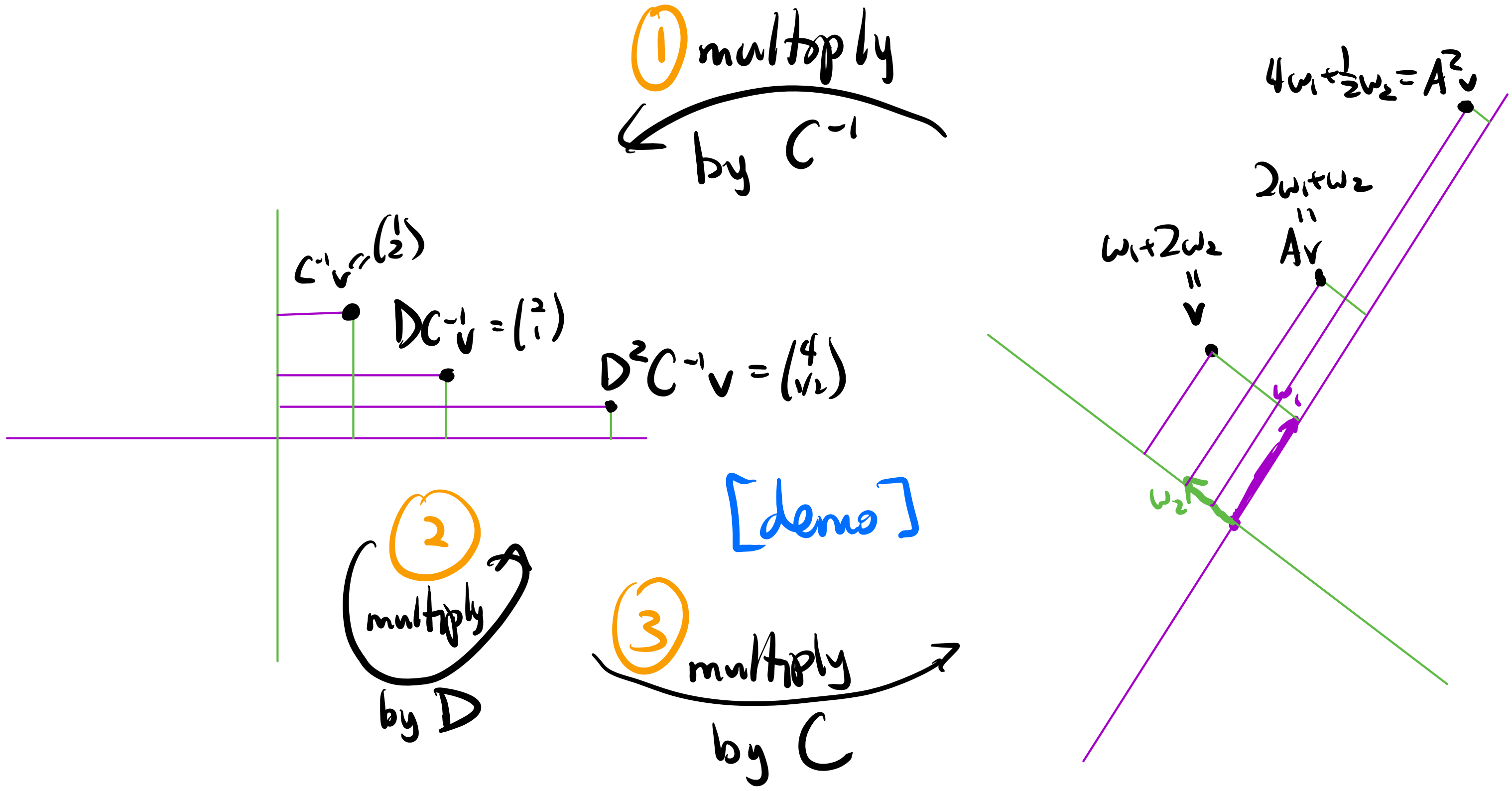
This is the vector form. In matrix form,

$$A = CDC^{-1} \qquad C = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Then $Av = CDC^{-1}v$

= " • first multiply v by C^{-1}
• then multiply by the diagonal matrix D
• then multiply by C again"

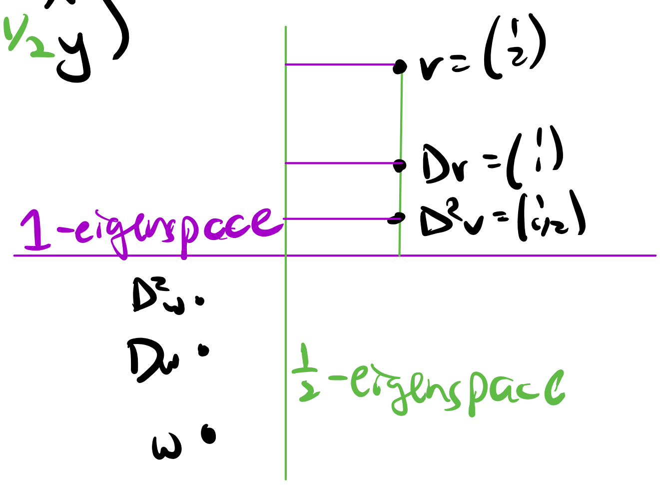
Note $C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 w_1 + x_2 w_2 \iff C^{-1}(x_1 w_1 + x_2 w_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



Eg: $D = \begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix}$ $\quad D\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1x\\ \frac{1}{2}y\end{pmatrix}$

- scales the x-direction by 1
- scales the y-direction by $\frac{1}{2}$

[demo]



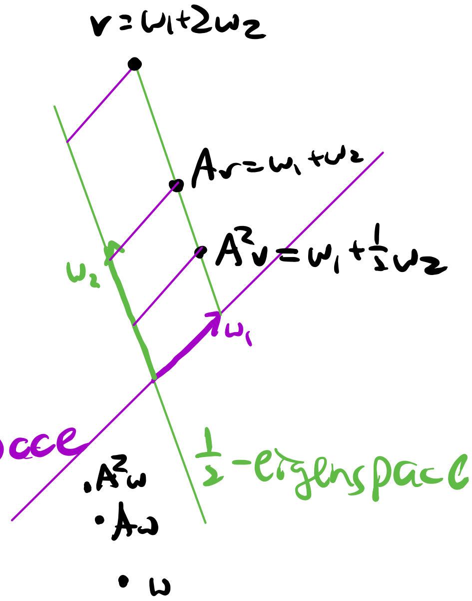
Eg: $A = \frac{1}{6}\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)(\lambda - \frac{1}{2})$

$\lambda_1 = 1 \quad w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = \frac{1}{2} \quad w_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Expand in the eigenbasis!

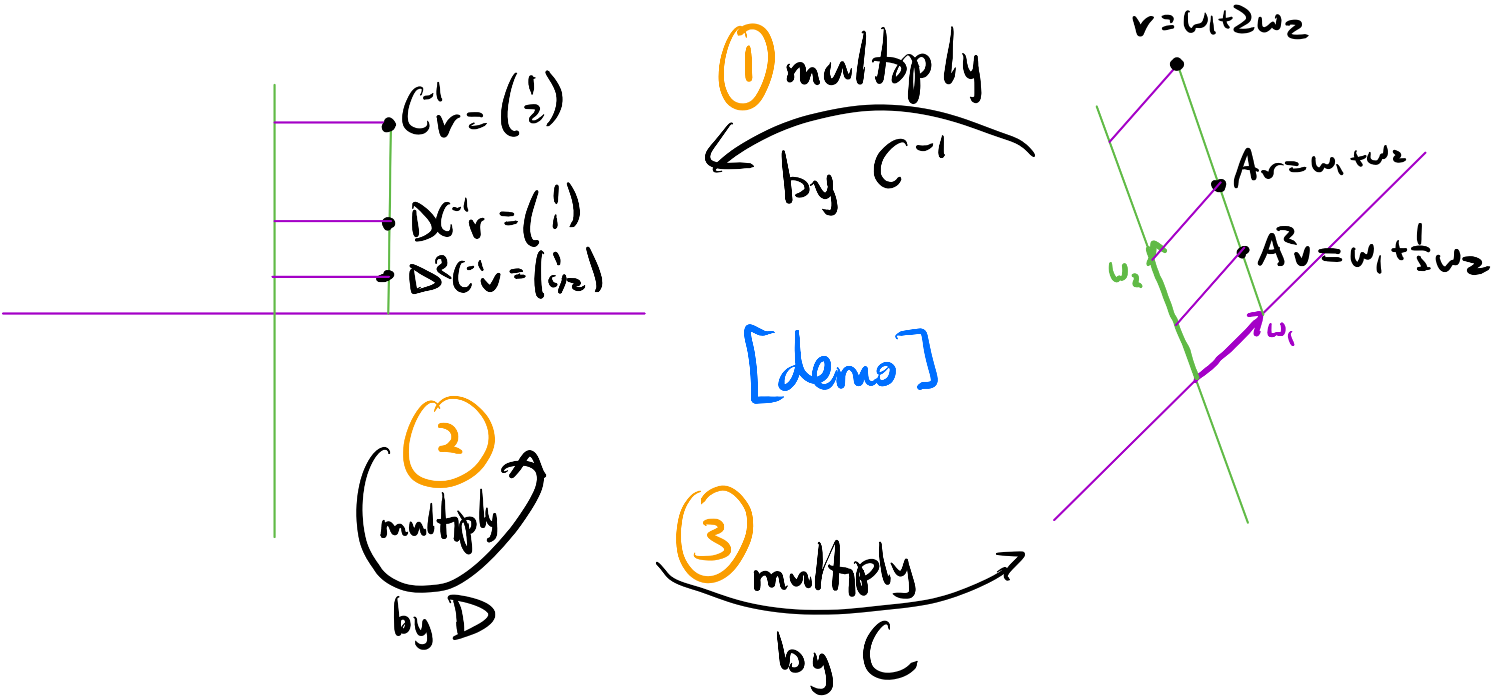
$$A(x_1 w_1 + x_2 w_2) = 1 x_1 w_1 + \frac{1}{2} x_2 w_2$$

- scales the w_1-direction by 1
- scales the w_2-direction by $1/2$



[demo]

Matrix Form: $A = CDC^{-1} \quad C = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$



Eg: $A = \frac{1}{580}\begin{pmatrix} 503 & 73 & 269 \\ 207 & 1137 & -49 \\ 270 & -30 & 680 \end{pmatrix}$ has eigenbasis

$$w_1 = \begin{pmatrix} -7 \\ 2 \\ 5 \end{pmatrix} \quad w_2 = \begin{pmatrix} -1 \\ -9 \\ 0 \end{pmatrix} \quad w_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

and eigenvalues

$$\lambda_1 = 1/2 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3/2$$

Expand in the eigenbasis!

$$A(x_1 w_1 + x_2 w_2 + x_3 w_3) = \frac{1}{2} x_1 w_1 + 2 x_2 w_2 + \frac{3}{2} x_3 w_3$$

- scales the w_1-direction by $\frac{1}{2}$
- scales the w_2-direction by 2
- scales the w_3-direction by $\frac{3}{2}$

[demo]