

Gaussian Elimination

This is how a computer solves systems of linear equations using elimination. Almost all questions in this class will reduce to this procedure! (The interesting part is how they do so.)

Def: Two matrices are row equivalent if you can get from one to the other using row operations.

NB: If augmented matrices are row equivalent then they have the same solution sets.

Algorithm (Gaussian Elimination/row reduction):

Input: Any matrix

Output: A row-equivalent matrix in REF.

Procedure:

(1a) If the first nonzero column has a zero entry at the top, row swap so that the top entry is nonzero.

$$\begin{bmatrix} 0 & 4 & 3 & 3 \\ 1 & 1 & -1 & 3 \\ 5 & -3 & -6 & -6 \end{bmatrix} \xrightarrow{R \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 5 & -3 & -6 & -6 \end{bmatrix}$$

This is now the first pivot position.

(1b) Perform row replacements to clear all entries below the first pivot.

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 5 & -3 & -6 & -6 \end{bmatrix} \xrightarrow{R_3 \leftarrow -5R_1} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & -8 & -1 & -21 \end{bmatrix}$$

Now **ignore** the row & column with the first pivot and **recurse** into the **submatrix** below and to the right:

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & -8 & -1 & -21 \end{bmatrix}$$

(2a) If the first nonzero column has a zero entry at the top, row swap so that the top entry is nonzero.

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & -8 & -1 & -21 \end{bmatrix} \quad \text{(Not applicable to this matrix)}$$

second pivot

(2b) Perform row replacements to clear all entries below the second pivot.

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & -8 & -1 & -21 \end{bmatrix} \xrightarrow{R_3 \leftarrow 2R_2} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & 5 & -15 \end{bmatrix}$$

etc. (recurse)

Doesn't mess up the 1st column!

In our example, the recursion has terminated:

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & 5 & -15 \end{bmatrix} \text{ is in REF!}$$

Important: If you want to apply this algorithm to an augmented matrix, just **delete** the augmentation line (pretend it's not augmented).

Demo: Gauss-Jordan slideshow

Use **Rabinoff's Reliable Row Reducer** on the HW!
↳ use the Sage cell when this gets easy

Jordan Substitution

This is the **back-substitution** procedure.

It is necessary when you have **∞ solutions**.

It puts a matrix into the following form:

Def: A matrix is in **reduced row echelon form (RREF)** if:

(1-2) It is in **REF**

← (so RREF implies REF)

(3) All pivots are equal to 1.

(4) A pivot is the only nonzero entry in its column.

REF

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\bullet = nonzero (pivot)

RREF

$$\begin{bmatrix} 1 & \bullet & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\bullet = any number

Eg:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & 0 & 10 & 30 \end{array} \right]$$

is in REF. How to put into RREF?
Do back substitution!

Row Operations

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & 0 & 10 & 30 \end{array} \right]$$

(scale so this is 1)

$$R_3 \div 10 \left\{ \begin{array}{l} \text{solve for } x_3 \end{array} \right.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

(kill these)

$$\begin{array}{l} R_1 \leftarrow 3R_3 \\ R_2 \leftarrow 10R_3 \end{array} \left\{ \begin{array}{l} \text{substitute } x_3 = 3 \text{ into } R_1 \& R_2 \\ \text{then move the constants to the RHS} \end{array} \right.$$

Back-Substitution

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -5x_2 - 10x_3 &= -20 \\ 10x_3 &= 30 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -5x_2 - 10x_3 &= -20 \\ x_3 &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 0 & 10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= -3 \\ -5x_2 &= 10 \\ x_3 &= 3 \end{aligned}$$

(scale so this is 1) $R_2 \div -5$ $\left\{ \begin{array}{l} \text{solve} \\ \text{for} \\ x_2 \end{array} \right.$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= -3 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

(kill this)

$R_1 \leftarrow R_1 - 2R_2$ $\left\{ \begin{array}{l} \text{substitute } x_2 = -2 \text{ into } R_1 \\ \text{then move the constants to} \\ \text{the RHS} \end{array} \right.$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

This is in RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\rightsquigarrow \begin{aligned} x_1 &= 1 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

Solved ✓

Upshot: Jordan substitution is exactly back-substitution.

Demo: Gauss-Jordan slideshow, cont'd

Algorithm (Jordan Substitution):

Input: A matrix in REF

Output: The row-equivalent matrix in RREF.

Procedure:

Loop, starting at the last pivot:

(a) Scale the pivot row so the pivot = 1.

(b) Use row replacements to kill the entries above that pivot.

"theorem"

Thm: The RREF of a matrix is unique.

In other words, if you start with a matrix, do any legal row operations at all, and end with a matrix in RREF, then it's the same matrix that Gauss-Jordan will produce.

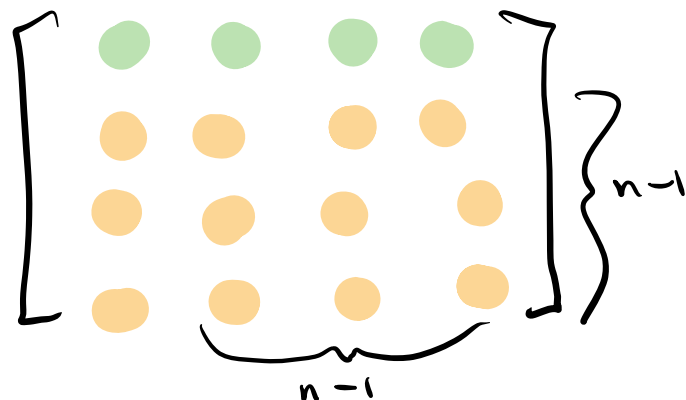
↳ Gaussian elimination + Jordan substitution.

NB: Jordan substitution gives you a RREF matrix with the same pivots. So uniqueness of RREF implies uniqueness of pivot positions.

Computational Complexity

How much computer time does Gauss-Jordan take?

Gaussian Elimination on an $n \times n$ matrix takes:



$$(n-1)(n-1) \text{ mult}$$

$$(n-1)(n-1) \text{ add}$$

$$2(n-1)^2$$

flops = floating point operations



$$(n-2)(n-2) \text{ mult} \\ + (n-2)(n-2) \text{ add} \\ \hline 2(n-2)^2 \text{ flops}$$

etc.

Step 1: Each row replacement requires $n-1$ multiplications & $n-1$ additions. (no computation in 1st col: just write "0")

Do for $n-1$ lower rows:

Step 2: Each row replacement requires $n-2$ multiplications & $n-2$ additions. Must do this for $n-2$ remaining rows

pyramidal number

$$\text{Total: } 2 \left[(n-1)^2 + (n-2)^2 + \dots + 1^2 \right]$$

$$= 2 \cdot \frac{n(n-1)(2n-1)}{6} \approx \frac{2}{3} n^3 \text{ flops}$$

Back-Substitution

$$\text{green } x_n = \text{brown}$$

1 mult = 1 flop

$$\text{purple } x_{n-1} + \text{green } x_n = \text{brown}$$

2 mult, 1 add = 3 flops
(substitute x_n , \times green, subtract, \div purple)

$$\text{red } x_{n-2} + \text{purple } x_{n-1} + \text{green } x_n = \text{brown}$$

3 mult, 2 add = 5 flops
(substitute x_n & x_{n-1} , \times green, \times purple, subtract, \div red)

$$\text{blue } x_1 + \dots + \text{green } x_n = \text{brown}$$

n mult, $(n-1)$ add = $2n-1$ flops

$$\text{Total: } 1 + 3 + 5 + \dots + (2n-1) = n^2 \text{ flops}$$

NB: $\frac{2}{3} n^3$ is a lot more than n^2 !

For a 1000×1000 matrix, $\frac{2}{3} n^3 \approx \frac{2}{3}$ gigaflops
but $n^2 = 1$ megaflop. If we want to solve
 $Ax=b$ for 1000 values of b , doing elimination
each time takes $\frac{2}{3}$ teraflops!

Inverse Matrices

Question: When solving $Ax=b$, when can we "divide by A "?

If " $x = \frac{b}{A}$ " makes sense, then $Ax=b$ has exactly one solution " $x = \frac{b}{A}$ " for every b .

This means $\text{RREF}(A|b)$ looks like this:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array} \right)$$

Def: An $n \times n$ (square!) matrix A is invertible if there exists another $n \times n$ matrix B such that $AB = I_n = BA$. $I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $n \times n$ identity matrix
Otherwise it's called singular.

Notn: $B = A^{-1}$, called the inverse of A .

Eg: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow B = A^{-1}$$

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{\text{"B"}} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq I_2$
so A is singular (non-invertible).

Remark: Since $AB \neq BA$ in general, you have to require $AB = I_n = BA$ a priori. **But:**

Fact: If A and B are $n \times n$ matrices and $AB = I_n$ or $BA = I_n$, then $B = A^{-1}$.

So the definition above is a bit pedantic...

Remark: A non-square matrix does not admit both a left- and right-inverse, so not invertible. (Can't solve $AB = I_m$ and $CA = I_n$ unless A is square.) This is why we only treat invertibility of **square** matrices.

Fact: $(A^{-1})^{-1} = A$

because $AB = I_n$ means $B = A^{-1}$ and $A = B^{-1}$

Fact: If A & B are invertible, then so is AB , and $(AB)^{-1} = \cancel{A^{-1}B^{-1}} B^{-1}A^{-1}$.

Check: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$ ✓

NB: Why not $(AB)^{-1} = A^{-1}B^{-1}$? Let's check:

$$(AB)(A^{-1}B^{-1}) = ABA^{-1}B^{-1} = ???$$

(no cancellation - can't re-order the terms!)

Thm: Let A be an $n \times n$ matrix. either all are true or all are false
The following are equivalent: (TFAE)

(1) A is invertible ← coefficient matrix!

(2) The RREF of A is I_n

(3) A has a pivot in every row/every column.
(A has n pivots)

We'll see why a bit later.

Eg: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ invertible
● = pivots

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is in RREF, $\neq I_2$ singular

How do you compute the inverse?

Algorithm (Matrix Inversion):

Input: A square matrix.

Output: The inverse matrix, or "singular"

Procedure:

(a) Form the augmented matrix $[A | I_n]$

(b) Run Gauss-Jordan on $[A | I_n]$.

(c) If the output is $[I_n | B]$ then $B = A^{-1}$.

Otherwise A is singular.

← why does this work?
Next time.

Eg: Compute $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \times = 2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array} \right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array} \right]$$

$$\xrightarrow{R_1 \div 2} \left[\begin{array}{cc|cc} 1/2 & 0 & 1 & -3/2 \\ 0 & 1 & -1 & 2 \end{array} \right]$$

This has the form $\left[I_2 \mid \begin{matrix} 2 & -3 \\ -1 & 2 \end{matrix} \right]$

$$\text{So } \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Eg: Compute $\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}^{-1}$.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array} \right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

But this is in RREF and it does not have the form $[I_2 \mid B] \Rightarrow$ singular.

NB: We knew this after the first step: no pivot in the second column.

Actually there's a **shortcut** for **2x2** matrices:

Fact: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad-bc \neq 0$,
in which case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eg: $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{2 \cdot 2 - 3 \cdot 1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Check: $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
 $= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & bd-bd \\ ac-ac & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

What is this good for?

Suppose A is invertible. Let's solve $Ax=b$.

$$Ax=b \iff A^{-1}(Ax)=A^{-1}b$$

$$\iff (A^{-1}A)x=A^{-1}b$$

$$\iff I_n x = A^{-1}b \iff x = A^{-1}b$$

For invertible A :

$$Ax=b \iff x=A^{-1}b$$

In particular, $Ax=b$ has exactly one solution for any b , and we have an expression for b in terms of x

Eg: Solve
$$\begin{aligned} 2x_1 + 3x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\iff \begin{aligned} x_1 &= 2b_1 - 3b_2 \\ x_2 &= -b_1 + 2b_2 \end{aligned}$$

So if you want to solve
$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

$$\Rightarrow x_1 = 2(3) - 3(4) = -6$$

$$x_2 = -(3) + 2(4) = 5$$