

Orientation: In some sense we've learned to solve a linear system $Ax=b$ using elimination & substitution. But there are some important questions still to answer:

(1) For which vectors b is $Ax=b$ consistent?
(ie for which b does a solution exist?)

(2) If a solution exists, how to describe **all** solutions of $Ax=b$?

Note that (2) is really only interesting if there are ∞ solutions.

Parametric Form

Now we deal systematically with systems of equations with ∞ solutions. We want to **parameterize** all solutions.

Eg:
$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array} \right]$$

RREF $\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array} \right] \rightsquigarrow \begin{aligned} x + 5z &= 1 \\ y + 2z &= -1 \end{aligned}$

Observation: If you substitute **any** number for z , you get the system

$$\begin{cases} x = 1 - 5z \\ y = -1 - 2z \end{cases}$$

Diagram: A bracket on the left groups the two equations. A purple arrow labeled "unknowns" points to x and y . An orange arrow labeled "numbers" points to the constants 1, -1, 5, and 2.

which has a unique solution!

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 5z \\ -1 - 2z \\ z \end{pmatrix} \quad \text{eg } z=1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$$

check:
$$\begin{aligned} 2(-4) + (-3) + 12(1) &= 1 \\ -4 + 2(-3) + 9(1) &= -1 \end{aligned}$$
 ✓

This is the **parametric form** of the solution;
 z is the **free variable** or **parameter**.

Implicit vs Parameterized Form.

- The system of equations
$$\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1 \end{cases}$$

are **implicit equations** of a line: it expresses the line as the set of **solutions** of these equations without giving you any way to write down specific points on the line.

(Good for **checking** if (a,b,c) is on the line)

- The parametrized form
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 5z \\ -1 - 2z \\ z \end{pmatrix}$$

is a **parametric equation** for the same line: it gives you a way to **produce** all solutions in terms of the **parameter** z .

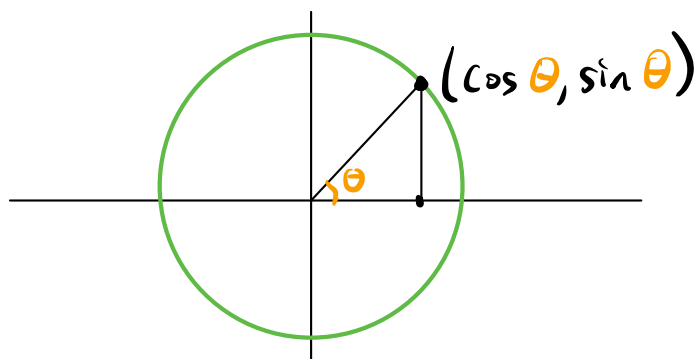
(Good for **producing** points on the line) [demo]

Non-linear example:

An **implicit equation** for the unit circle is

$$x^2 + y^2 = 1$$

A **parametric equation** for the unit circle is



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \theta = \text{parameter}$$

Here's how to produce parametric equations for general linear systems.

Recall: A **pivot column** of a matrix is a column with a pivot.

Def: A **free variable** in a system of equations is a variable whose column (in the coeff matrix) is **not** a pivot column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array} \right]$$

$x \quad y \quad z$

 x, y in pivot cols
 z is **free**

These are the variables you can't **isolate** in back-substitution.

Procedure (Parametric Form):

To find the **parametric form** of the solutions of $Ax=b$:

(1) Put $[A|b]$ into **RREF**. Stop if inconsistent.

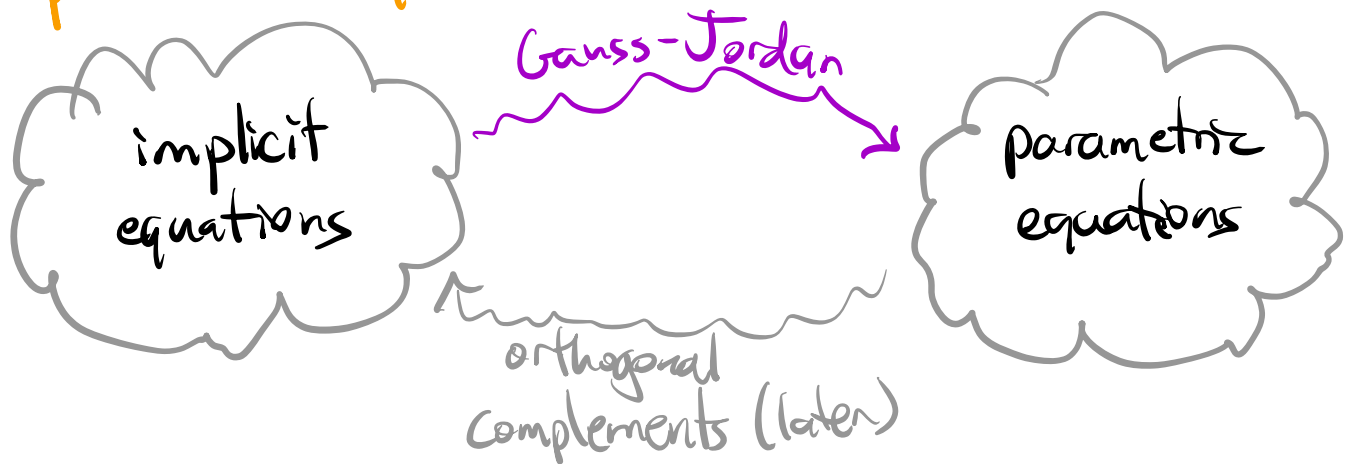
(2) Write out the corresponding equations

(3) **Move free variables to the right-hand side**

All solutions are obtained by substituting **any values** for the free variables.

This uses the free variables as the **parameters**.

So Gauss-Jordan elimination turns implicit equations into parametric equations. ↳ (step (i) above)



Eg: $x + y + z = 1 \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 \end{bmatrix}$

this is already in RREF!

Free variables: y, z

$\xrightarrow[\text{form}]{\text{parametric}}$ $x = 1 - y - z$

This is a parameterized plane. [demo]

Eg: $\begin{matrix} x + y = 2 \\ x - y = 0 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & | & 2 \\ 1 & -1 & | & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \end{bmatrix}$

No free variables! Just have one solution
 $x = 1, y = 1.$

Observation:

- 2 free variables / 2 parameters:
solution set is a plane
- 1 free variable / 1 parameter:
solution set is a line
- 0 free variables / 0 parameters:
solution set is a point

Provisional Defⁿ: The dimension of the solution set of a consistent system $Ax=b$ is the number of free variables.

Parametric Vector Form

This is an alternate, more concise way of writing a solution set in parametric form.

Eg: $2x + y + 12z = 1$
 $x + 2y + 9z = -1$

parametric
form

$$\begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \\ z &= z \end{aligned}$$

write in columns
(from before)

parameterize the free variable too

Let's rewrite this as one equation involving vectors:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

linear combination of $(-5, -2, 1)$

This is the line thru $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ in the $\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$ -direction.

[demo again]

Eg: $x + y + z = 1$

parametric
form

$$\begin{aligned} x &= 1 - y - z \\ y &= y \\ z &= z \end{aligned}$$

write in columns

parameterize the free variables

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

linear combination of $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

This is the plane containing $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, & $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

[demo again]

Writing the solution set in this way is called the **parametric vector form**.

Procedure (Parametric Vector Form)

To find the **parametric vector form** of the solutions of $Ax=b$:

(1-3) Find the parametric form

(4) Add trivial equations for the free variables, in order. Organize the right-hand side into columns.

(5) Gather the columns into vectors. Pull out the free variables as coefficients.

Result:

$$X = \left(\begin{array}{c} \text{a constant} \\ \text{vector} \end{array} \right) + \left(\begin{array}{c} \text{a linear combination with} \\ \text{the free variables as weights} \end{array} \right)$$


NB: The constant vector is the solution you get by setting all free variables = 0.

Def: This vector is called a **particular solution**.
(It is a solution of $Ax=b$)

Eg:
$$\begin{aligned} x + 2y + 2z + w &= 1 \\ 2x + 4y + z - w &= -1 \end{aligned} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 1 & -1 & -1 \end{array} \right]$$

RREF
$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

$$\begin{aligned} x + 2y - w &= -1 \\ z + w &= 1 \end{aligned}$$

free

$$\begin{cases} x = -1 - 2y + w \\ y = y \\ z = 1 - w \\ w = w \end{cases}$$

← trivial equations
↑ columns

PVF
$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

↑ particular solution
any linear combination of $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Vector Equations

This is another way of writing a linear system that works well with what we've been doing.

Def: A **vector equation** is an equation of the form

$$x_1 v_1 + x_2 v_2 + \dots + x_n v_n = b \quad v_1, \dots, v_n, b \in \mathbb{R}^m$$

for **unknown scalars** $x_1, \dots, x_n$.

This is:

$$\left(\begin{array}{l} \text{linear combination of} \\ v_1, \dots, v_n \text{ with } \text{unknown weights} \end{array} \right) = (\text{vector})$$

Eg: $x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$

This is equivalent to the system

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(use the column-first definition of the matrix-vector product). But now we're thinking in terms of linear combinations of vectors.

→ we will draw **pictures** of these (next time)

Four Ways to Write a System of Eqs:

(1) Linear system

$$\begin{aligned}x_1 - x_2 &= 8 \\ 2x_1 - 2x_2 &= 16 \\ 6x_1 - x_2 &= 3\end{aligned}$$

(2) Matrix Equation

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(columns)

(3) Augmented Matrix

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array} \right)$$

(4) Vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

You still **solve** a vector equation by putting it into an augmented matrix:

$$\text{Eg: } x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array} \right]$$
$$\xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array} \right]$$

Solution is $x_1 = -1, x_2 = -9$

Important Observation: **(!!!!)**

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \text{ has a solution (consistent)}$$

$$\Leftrightarrow \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \text{ is a linear combination of } \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

in which case the solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the vector of **weights**

In fact, we know $x_1 = -1$, $x_2 = -9$:

$$-1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} = \text{(linear combination of } \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \text{ \& } \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix})$$

different b
↓

Eg: $x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & -2 & -2 \\ 6 & -1 & 0 \end{array} \right]$

REF $\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 5 & -12 \\ 0 & 0 & -6 \end{array} \right]$ inconsistent

So $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$ is **not** a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$.

In general:

next time
↓

Column Picture Criterion for Consistency

$Ax = b$ is **consistent** (has at least one solution)



b is a **linear combination** of the **columns** of A
in which case $x =$ the weights.

We now have 2 ways that linear combinations appear when solving a system of equations:

Linear Systems & Linear Combinations

(1) $Ax=b$ is consistent $\iff b$ is a linear combination of the columns of A .

(2) In this case, the solution set has the form
$$x = \begin{pmatrix} \text{particular} \\ \text{solution} \end{pmatrix} + \begin{pmatrix} \text{all linear combinations} \\ \text{of a set of vectors} \end{pmatrix}$$

Next time: Spans: this is what the set of all linear combinations of a list of vectors looks like.