# Complex Numbers: Crash Course

**Eg:** Diagonalize $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (CCW rotation by $90^\circ$)

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 + 1$$

This has no real roots: $\lambda^2 + 1 = 0 \Leftrightarrow \lambda^2 = -1$.

**Solution:** Add a $\sqrt{-1}$ to our number system!

**Def:** The unit imaginary number is a number $i$ such that $i^2 = -1$. A complex number is a number $a+bi$ for $a, b \in \mathbb{R}$.

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

is the set of all complex numbers.

If $z = a + bi$ is a complex number, its

- real part is $\mathrm{Re}(z) = a$, and its
- imaginary part is $\mathrm{Im}(z) = b$.

We can add/subtract & multiply complex numbers:

$$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$$

$$(a+bi)(c+di) = ac + adi + bci + bd\underbrace{i^2}_{=-1}$$

$$= (ac - bd) + (ad + bc)i$$

Division: see p.4

Aside

**Question:** Wait! Why can I just declare that $-1$ has a square root?

**Answer 1:** Why can you declare that $2$ has a square root? You can't write it down—it's an infinite non-repeating decimal...

**Answer 2:** Take Math 401/501 for a systematic treatment.

Complex numbers have an additional algebraic operation.

**Def:** The **complex conjugate** of $z = a+bi$ is

$$\bar{z} = a - bi$$

(replace $i$ by $-i$ = the other $\sqrt{-1}$)

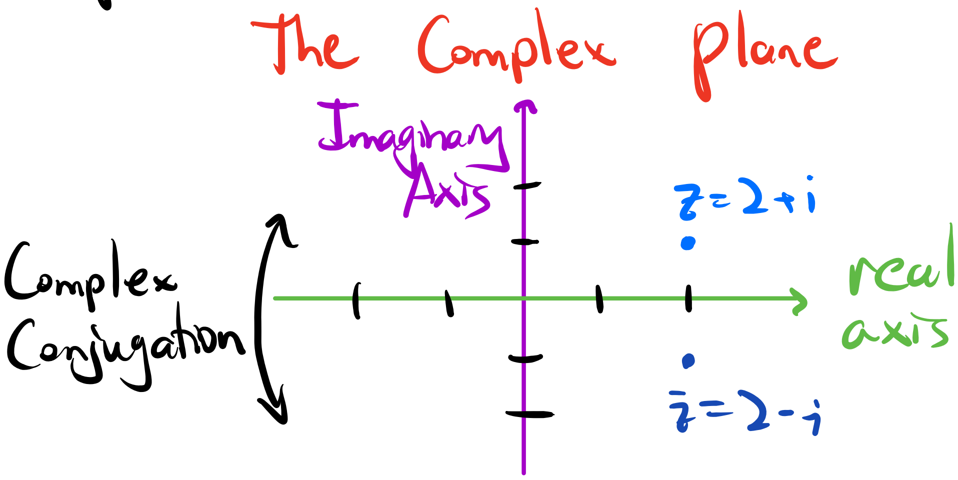
**Check:** $\overline{(z+w)} = \bar{z} + \bar{w}$ $\quad \overline{zw} = \bar{z}\cdot\bar{w}$ $\quad \bar{\bar{z}} = z$

**NB:** if $z = a+bi$ then

- $z + \bar{z} = (a+bi) + (a-bi) = 2a = 2\,\mathrm{Re}(z)$
- $z - \bar{z} = (a+bi) - (a-bi) = 2bi = 2i\,\mathrm{Im}(z)$

$$\boxed{z + \bar{z} = 2\,\mathrm{Re}(z) \qquad z - \bar{z} = 2i\,\mathrm{Im}(z)}$$

Since a complex number $z = a+bi$ is determined by two real numbers $a$ and $b$, we can draw $\mathbb{C}$ as a plane:




Complex conjugation negates the imaginary coordinate: it flips over the real axis.

NB: A real number is also a complex number:

$$a \in \mathbb{R} \rightsquigarrow a + 0i \in \mathbb{C}$$

So $\mathbb{C}$ contains $\mathbb{R}$. NB: $\boxed{z \in \mathbb{R} \Longleftrightarrow z = \bar{z}}$

NB: If $z = a+bi$ then

$$z\bar{z} = (a+bi)(a-bi) = a^2 - b i^2 = a^2 + b^2$$

nonnegative real number

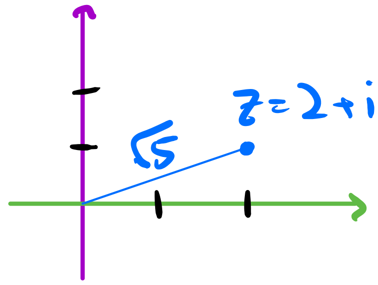
$$\boxed{z = a+bi \rightsquigarrow z\bar{z} = a^2 + b^2 \geq 0}$$

**Def:** The modulus of $z$ is $|z| = \sqrt{z\bar{z}}$

This is its length as a vector in the complex plane.

Eg: $z = 2+i$

$\rightsquigarrow |z| = \sqrt{4+1} = \sqrt{5}$




Eg: If $a \in \mathbb{R}$ then $a = a + 0i$ and

$$|a| = \sqrt{a^2+0^2} = \sqrt{a^2} = |a|$$

(the modulus is the usual absolute value)

Check: $|zw| = |z| \cdot |w|$ $\quad |\bar{z}| = |z|$

Here's how to take a reciprocal of $z = a+bi \neq 0$:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

positive real number

$$\boxed{\frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad \text{or} \quad \frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i}$$

Check: $\left|\frac{1}{z}\right| = \frac{1}{|z|}$

Eg: $\frac{1}{2+i} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5} i$

Check: $(2+i)\left(\frac{2}{5} - \frac{1}{5} i\right) = \frac{4}{5} + \frac{1}{5} + \left(\frac{2}{5} - \frac{2}{5}\right) i = 1$ ✓

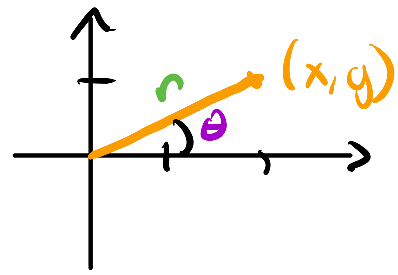
Now we can divide too: $\frac{w}{z} = w \cdot \frac{1}{z} = \frac{w\bar{z}}{|z|^2}$.

# Polar Coordinates

Recall that a point in the $(x,y)$-plane can be specified in polar coordinates $(r,\theta)$:

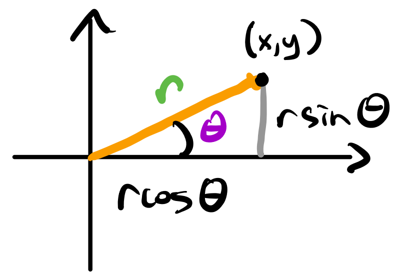
$r$ = length of $\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{x^2+y^2}$

$\theta$ = angle $\begin{pmatrix} x \\ y \end{pmatrix}$ makes with the positive x-axis $= \pm\arctan\left(\frac{y}{x}\right)$




To go from polar coordinates back to Cartesian $(x,y)$-coordinates:

$$\boxed{(r,\theta) \rightsquigarrow \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}}$$




If we apply this to a complex number $z=a+bi$:

$r=\sqrt{a^2+b^2}=|z| \rightsquigarrow a=|z|\cos\theta \quad b=|z|\sin\theta$

$$\Longrightarrow \boxed{z=|z|(\cos\theta+i\sin\theta)}$$

**Def:** The argument of $z=a+bi$ is

$\arg(z)=\theta=$ the angle $z$ makes with the positive real axis.

So you can specify a complex number in 2 ways:

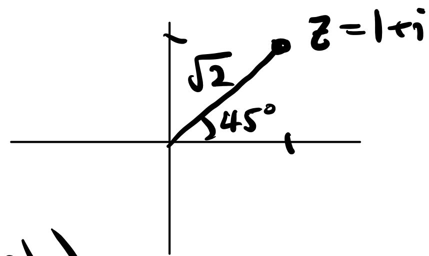
(Cartesian coords / real & imaginary parts)

$$z = a + bi$$

(Polar coords / modulus & argument)

$$z = |z|(\cos\theta + i\sin(\theta)) \qquad \theta = \arg(z)$$

Eg: $z = 1 + i \rightsquigarrow |z| = \sqrt{1+1} = \sqrt{2}$



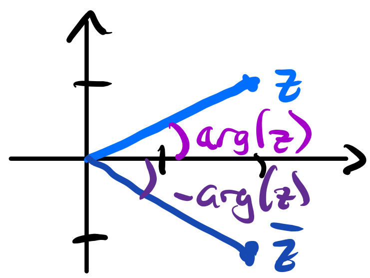

$\arg(z) = 45°$

$\rightsquigarrow \quad z = \sqrt{2}\,(\cos(45°) + i\sin(45°))$

Check: $\cos(45°) = \sin(45°) = 1/\sqrt{2}$

$z = 1 + i = \sqrt{2}\,(1/\sqrt{2} + i/\sqrt{2})$ ✓

Facts:




- $\arg(\bar{z}) = -\arg(z)$

  (flip over real axis)

- $\arg(1/z) = -\arg(z)$

  $\left( 1/z = \bar{z}/|z|^2 = (\text{positive number}) \cdot \bar{z} \right)$

**Fact:** $\arg(zw) = \arg(z) + \arg(w)$

**Proof:** This is a trig identity!

$$z = |z|(\cos\theta + i\sin\theta) \qquad \theta = \arg(z)$$
$$w = |w|(\cos\varphi + i\sin\varphi) \qquad \varphi = \arg(w)$$
$$zw = |z|\cdot|w|(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$$
$$= |zw|\left(\cos\theta\cos\varphi - \sin\theta\sin\varphi + (\cos\theta\sin\varphi + \sin\theta\cos\theta)i\right)$$

sum angle formulas

$$= |zw|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$$
$$\Rightarrow \theta + \varphi = \arg(zw)$$

We like polar coordinates because multiplication is easier!

$$z = |z|(\cos\theta + i\sin\theta) \qquad w = |w|(\cos\varphi + i\sin\varphi)$$
$$\rightsquigarrow zw = |z||w|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$$
ie. $|zw| = |z||w|$, $\arg(zw) = \arg(z) + \arg(w)$

$$\bar{z} = |z|(\cos\theta - i\sin\theta) \qquad \frac{1}{z} = \frac{1}{|z|}(\cos\theta - i\sin\theta)$$

**Exercise:** expand the product
$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$
to derive the triple-angle formulas.

**Euler's Formula:** For any real number $\theta$,

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

**Proof:** Take Taylor expansions of both sides...

**Polar Form of Complex Numbers, Alternative:**

$$\boxed{z = |z|e^{i\theta} \qquad \theta = \arg(z)}$$

This makes the formulas on p. 7 easier to remember:

$$zw = (|z|e^{i\theta})(|w|e^{i\varphi}) = |z|\cdot|w|e^{i(\theta+\varphi)}$$

$$\bar{z} = |z|e^{-i\theta}$$

$$\frac{1}{z} = (|z|e^{i\theta})^{-1} = \frac{1}{|z|}e^{-i\theta}$$

**Eg:** $-1 = e^{i\pi}$ $\qquad 1+i = \sqrt{2}e^{i\pi/4}$ (cf. p.6)

You can exponentiate any complex number:

$$\boxed{e^{a+bi} = e^a e^{bi} = e^a(\cos b + i\sin b)}$$

Since $\sin(-b) = -\sin b$, we have

$$\boxed{e^{\bar{z}} = \overline{e^z} \quad \text{for any } z \in \mathbb{C}}$$

It turns out that once $x^2+1=0$ has a solution, then any polynomial has a root!

**Fundamental Theorem of Algebra:**

Every polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$$a_0, a_1, \ldots, a_n \in \mathbb{C} \qquad a_n \neq 0$$

can be factored into linear terms:

$$p(x) = a_n(x-\lambda_1)\cdots(x-\lambda_n) \qquad \lambda_1, \ldots, \lambda_n \in \mathbb{C}$$

**Eg:** $p(x) = x^2+x+1$

Use the quadratic formula:

$$x = \frac{1}{2}\left(-1 \pm \sqrt{1-4}\right) = \frac{1}{2}\left(-1 \pm i\sqrt{3}\right)$$

$$\sqrt{-3} = \sqrt{-1}\cdot\sqrt{3} = i\sqrt{3}$$

So $p(x) = \left(x - \frac{1}{2}(-1+i\sqrt{3})\right)\left(x - \frac{1}{2}(-1-i\sqrt{3})\right)$

**Eg:** $p(x) = x^2+1 = (x+i)(x-i)$

So now $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has two eigenvalues $\pm i$

so it's diagonalizable!

## Real Polynomials:

if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has real coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ then its complex roots come in conjugate pairs:

$$p(\lambda) = 0 \iff p(\bar{\lambda}) = 0$$

## Check:

$$p(\bar{\lambda}) = a_n \bar{\lambda}^n + a_{n-1}\bar{\lambda}^{n-1} + \cdots + a_1\bar{\lambda} + a_0$$

$$(\bar{a}_i = a_i) \quad = \overline{a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0}$$

$$= \overline{p(\lambda)}$$

So $p(\lambda) = 0 \iff p(\bar{\lambda}) = 0.$

**Eg:** The roots of $p(x) = x^2 + x + 1$ are

$$\lambda = \tfrac{1}{2}(-1 + i\sqrt{3}) \quad \text{and} \quad \bar{\lambda} = \tfrac{1}{2}(-1 - i\sqrt{3})$$