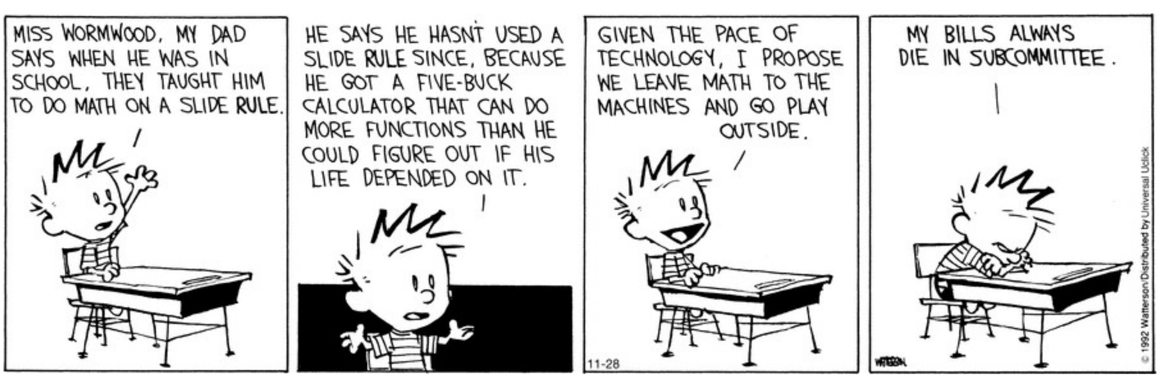
MATH 218D-1
MIDTERM EXAMINATION 1

Name		Duke NetID	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



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Problem 1. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 4 & -2 \end{pmatrix}.$$

a) Find the parametric vector form of the solution set of $Ax = \binom{2}{6}$.

$$x = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} +$$

b) Two different solutions of $Ax = \binom{2}{6}$ are:

$$x = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

c) The solution set of $Ax = \binom{2}{6}$ is a (○point ○line ○plane ○space) in $\mathbf{R}^{\square}$.

d) Find a basis for Nul(A).

$$\left\{ \qquad \right\}$$

[Scratch work for Problem 1]

e) Find a basis for Col(A).

$$\left\{ \qquad \qquad \qquad \right\}$$

f) Is the vector $(1, 2, 3, 4)$ in Row(A)? ◯ Yes ◯ No

g) Find a vector $b \in \mathbf{R}^2$ such that $Ax = b$ has solution set

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \operatorname{Span}\left\{ \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\},$$

or explain why no such vector exists.

h) Find a vector $b \in \mathbf{R}^2$ such that $Ax = b$ is inconsistent, or explain why no such vector exists.

[Scratch work for Problem 1]

Problem 2.

[15 points]

Consider the subspace

$$V = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\}.$$

a) Find a basis for $V^{\perp}$.

$$\left\{ \qquad\qquad \right\}$$

b) $\dim(V^{\perp}) = \boxed{}$.

c) $\dim(V) = \boxed{}$.

d) Which of the following sets form a basis for V? Fill in the bubbles of all that apply.

○ $\left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\}$

○ $\left\{ \begin{pmatrix} 1 \\ -3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ -9 \\ 10 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix} \right\}$ ○ $\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

[Scratch work for Problem 2]

Problem 3. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 5 & 6 \\ -2 & -4 & 4 \end{pmatrix}.$$

a) Compute A^{-1} using Gauss–Jordan elimination. Please write out all row operations that you perform. (You can check your work by multiplying your answer by A.)

$$A^{-1} = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

b) Express A^{-1} as a product of elementary matrices.

$A^{-1} =$

c) In the $A = LU$ decomposition of A, compute the matrix U, and write L as a product of elementary matrices. (You do not need to compute L.)

$$U = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \qquad L =$$

[Scratch work for Problem 3]

(Problem 3, continued)

Now we switch matrices to avoid carry-through error. Consider the matrix B with LU decomposition

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -3 \end{pmatrix}.$$

d) Solve $Bx = (2, -2, 1)$.

$$x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 3]

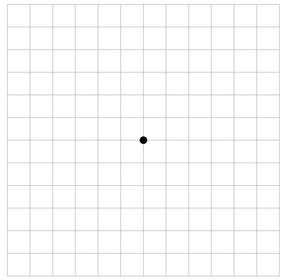
Problem 4. [10 points]

a) Write a 2×2 matrix A of rank 2.

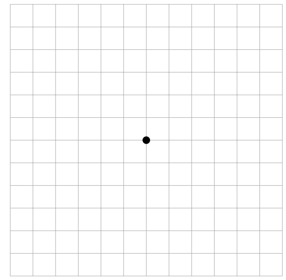
b) Draw the null space of A in the grid on the left and the column space of A in the grid on the right. Be precise!

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Nul(A)



Col(A)

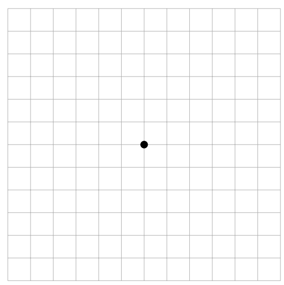


c) Write a 2×2 matrix B of rank 1.

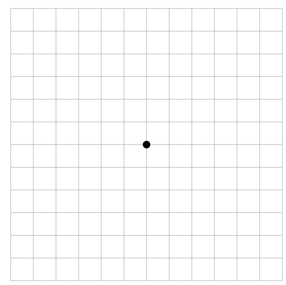
d) Draw the null space of B in the grid on the left and the column space of B in the grid on the right. Be precise!

$$B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Nul(B)



Col(B)

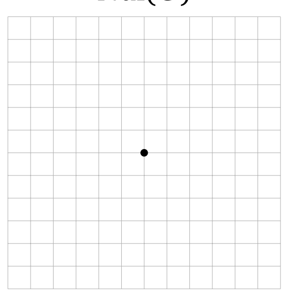


e) Write a 2×2 matrix C of rank 0.

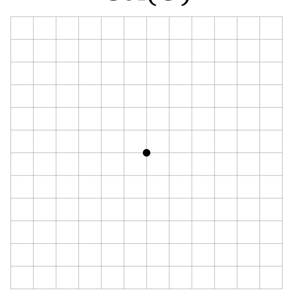
f) Draw the null space of C in the grid on the left and the column space of C in the grid on the right. Be precise!

$$C = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Nul(C)



Col(C)



[Scratch work for Problem 4]

Problem 5.

[20 points]

Short-answer questions: no justification is necessary unless indicated otherwise.

a) If A is a 200×250 matrix then

$$\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A) = \boxed{}.$$

b) For a certain 4×5 matrix A, the matrix equation

$$Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{has solution set} \quad x = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 4 \\ 1 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Which of the following can you conclude about A? Fill in the bubbles of all that apply.

- ◯ $\mathrm{rank}(A) = 4$.
- ◯ $\mathrm{Nul}(A)$ is a line in $\mathbf{R}^5$.
- ◯ A has full row rank.
- ◯ $A(1, 0, -1, 0, 1) = 0$.
- ◯ $Ax = b$ has infinitely many solutions for all $b \in \mathbf{R}^4$.
- ◯ A has full column rank.
- ◯ $\mathrm{Col}(A) = \mathbf{R}^4$.
- ◯ The rows of A are linearly independent.
- ◯ $\mathrm{Row}(A) = \mathbf{R}^4$.
- ◯ $Ax = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$ has solution set $x = \begin{pmatrix} -3 \\ 0 \\ -2 \\ -4 \\ -1 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$.

c) Consider the vectors

$$\left\{ \begin{pmatrix} 1 \\ 7 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 9 \\ 11 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}.$$

These vectors are (choose one):

- ◯ linearly independent.
- ◯ linearly dependent.
- ◯ not enough information to tell.

d) Which of the following subspaces are equal to $V = \{(x, y, z) \in \mathbf{R}^3 \ : \ x = z\}$? Fill in the bubbles of all that apply.

- ◯ $\mathrm{Nul}\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$
- ◯ $\mathrm{Col}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$
- ◯ $\mathrm{Span}\{(1, 0, -1)\}^{\perp}$
- ◯ $\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$
- ◯ $\{(x, y, x) \ : \ x, y \in \mathbf{R}\}$
- ◯ $\mathrm{Row}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 0 & 3 \end{pmatrix}$

[Scratch work for Problem 5]

Problem 6. [20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A matrix that has full column rank but is not invertible.

b) A 2×2 matrix that does not have an LU decomposition.

c) A 3×3 matrix whose columns are linearly independent but whose rows are linearly dependent.

d) A 2×2 matrix A such that $Ax = \binom{1}{2}$ has a solution but $Ax = \binom{2}{1}$ does not.

e) A matrix A such that $A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and $A^T\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

[Scratch work for Problem 6]