Math 218D-1: Homework #1

Answer Key

1. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u+v+w$ and $u+2v-w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Explain why every linear combination of u, v, w is also a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to _____?

e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

Solution.

a) $u+v+w = \begin{pmatrix} 6 \\ -7 \\ 1 \end{pmatrix} \qquad u+2v-w = \begin{pmatrix} -13 \\ 6 \\ 7 \end{pmatrix}$

b) $w = 2u - 2v$

c) Substitute $w = 2u - 2v$ in a linear combination.

d) Zero.

e) Anything whose coordinates do not sum to zero.

2. Find two *different* triples (x, y, z) such that

$$x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

How many such triples are there?

Solution.

Choose any value of z, and set

$$\begin{aligned} x &= 2 - \tfrac{5}{4}z \\ y &= 2 - \tfrac{3}{4}z. \end{aligned}$$

3. Decide if each statement is true or false, and explain why.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

c) If v, w are two vectors in $\mathbf{R}^2$, then any other vector b in $\mathbf{R}^2$ is a linear combination of v and w.

Solution.

a) True.

b) False.

c) False (suppose v and w are parallel).

4. Suppose that v and w are *unit vectors*: that is, $v \cdot v = 1$ and $w \cdot w = 1$. Compute the following dot products using the algebra of dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v+w)\cdot(v-w)$ **c)** $(v+2w)\cdot(v-2w)$.

Solution.

a) $v \cdot (-v) = -(v \cdot v) = -1$

b) $(v+w)\cdot(v-w) = v \cdot v - w \cdot w = 0$

c) $(v+2w)\cdot(v-2w) = v\cdot v - 2^2 w \cdot w = -3$

5. Two vectors v and w are *orthogonal* if $v \cdot w = 0$. Find nonzero vectors v and w in $\mathbf{R}^3$ that are orthogonal to $(1,1,1)$ and to each other.

Solution.

There are many answers. For instance, $v = (1,-1,0)$ and $w = (1,1,-2)$.

6. Compute the following matrix-vector products using *both* the by-row and by-column methods. If the product is not defined, explain why.

$$\begin{pmatrix}2\\5\end{pmatrix}\begin{pmatrix}1\\-3\\-1\end{pmatrix} \qquad \begin{pmatrix}1&-2\\0&-1\\3&2\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix} \qquad \begin{pmatrix}7&2&4\\3&-3&1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}7&4\\-2&2\\4&1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} \qquad \begin{pmatrix}2&6&-1\end{pmatrix}\begin{pmatrix}5\\-1\\0\end{pmatrix} \qquad \begin{pmatrix}5\\-1\\0\end{pmatrix}\begin{pmatrix}2&6&-1\end{pmatrix}$$

Solution.

$$[\text{undefined}] \qquad [\text{undefined}] \qquad \begin{pmatrix}9\\7\end{pmatrix}$$

$$\begin{pmatrix}-1\\-6\\2\end{pmatrix} \qquad \begin{pmatrix}4\end{pmatrix} \qquad \begin{pmatrix}10&30&-5\\-2&-6&1\\0&0&0\end{pmatrix}$$

7. Suppose that $u = (x, y, z)$ and $v = (a, b, c)$ are vectors satisfying $2u + 3v = 0$. Find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution.
$w = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

8. Consider the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad E = \begin{pmatrix} -3 & 5 \end{pmatrix}.$$

Compute the following expressions. If the result is not defined, explain why.

a) $-3A$ **b)** $B - 3A$ **c)** AC **d)** B^2

e) $A + 2B$ **f)** $C - E$ **g)** EB **h)** D^2

Solution.

a) $\begin{pmatrix} -6 & -3 & 3 \\ -12 & 12 & -6 \end{pmatrix}$

b) $\begin{pmatrix} -1 & 0 & 5 \\ -11 & 11 & -4 \end{pmatrix}$

c) [undefined]

d) [undefined]

e) $\begin{pmatrix} 12 & 7 & 3 \\ 6 & -6 & 6 \end{pmatrix}$

f) [undefined]

g) $\begin{pmatrix} -10 & -14 & 4 \end{pmatrix}$

h) $\begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}$

9. Compute the product

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

in two ways:

a) using the column form, and

b) using the outer product form.

Solution.

Both give the same answer:

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 10 & -1 & 3 \\ 0 & 3 & -4 \end{pmatrix}.$$

10. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & h \end{pmatrix}.$$

What value(s) of h, if any, will make $AB = BA$?

Solution.

$h = 1$

11. Consider the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} -4 & -8 \\ 5 & 8 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Verify that $AC = BC$ and yet $A \neq B$.

Solution.

$AC = \begin{pmatrix} -4 & -4 \\ 3 & 3 \end{pmatrix} = BC$

12. Show that $(A+B)^2 \neq A^2 + 2AB + B^2$ when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

What is the correct formula?

$$(A+B)^2 = A^2 + B^2 + \underline{\qquad\quad}$$

[**Hint:** distribute the product $(A+B)(A+B)$.]

Solution.

In this example,

$$(A+B)^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad A^2 + 2AB + B^2 = \begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix}.$$

The correct formula is

$$(A+B)^2 = A^2 + AB + BA + B^2.$$

13. In the following, find the 2×2 matrix A that acts in the specified manner.

a) $A\binom{x}{y} = \binom{x}{y}$: the identity matrix does not change the vector.

b) $A\binom{x}{y} = \binom{y}{x}$: this is a flip over the line $y = x$.

c) $A\binom{x}{y} = \binom{y}{-x}$: this rotates vectors clockwise by 90°.

d) $A\binom{x}{y} = -\binom{x}{y}$: this rotates vectors by 180°.

e) $A\binom{x}{y} = \binom{0}{y}$: this projects onto the y-axis.

f) $A\binom{x}{y} = \binom{x}{0}$: this projects onto the x-axis.

g) $A\binom{x}{y} = \binom{x}{y-2x}$: this performs the row operation $R_2 -= 2R_1$.

[**Hint:** compute $A\binom{1}{0}$ and $A\binom{0}{1}$.]

Solution.

$$\textbf{a)}\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad \textbf{b)}\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \textbf{c)}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad \textbf{d)}-\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$\textbf{e)}\begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \textbf{f)}\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \textbf{g)}\begin{pmatrix}1&0\\-2&1\end{pmatrix}$$

14. Consider the matrices

$$A = \begin{pmatrix}1&1&1\\1&2&3\\1&4&5\end{pmatrix}\qquad D = \begin{pmatrix}2&0&0\\0&3&0\\0&0&5\end{pmatrix}.$$

Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by the diagonal matrix D on the right or the left.

Solution.

We have

$$AD = \begin{pmatrix}2&3&5\\2&6&15\\2&12&25\end{pmatrix}\qquad DA = \begin{pmatrix}2&2&2\\3&6&9\\5&20&25\end{pmatrix}.$$

Multiplying on the right by D scales the *columns* of A by the diagonal entries of D; multiplying on the left by D scales the *rows*.

15. Let A be a 4×3 matrix satisfying

$$Ae_1 = \begin{pmatrix}1\\3\\2\\7\end{pmatrix}\qquad Ae_2 = \begin{pmatrix}4\\4\\-1\\-1\end{pmatrix}\qquad Ae_3 = \begin{pmatrix}0\\0\\-2\\1\end{pmatrix}.$$

What is A?

Solution.

We know that Ae_i is the ith column of A, so

$$A = \begin{pmatrix}1&4&0\\3&4&0\\2&-1&-2\\7&-1&1\end{pmatrix}.$$

16. Suppose that A is a 4×3 matrix such that

$$A\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}7\\1\\2\\9\end{pmatrix} \qquad A\begin{pmatrix}0\\1\\4\end{pmatrix} = \begin{pmatrix}1\\-4\\2\\3\end{pmatrix}.$$

Compute Ax, where x is the vector $(3,-2,-2)$.
[**Hint:** express $(3,-2,-2)$ as a linear combination of $(1,0,2)$ and $(0,1,4)$.]

Solution.

We have

$$x = \begin{pmatrix}3\\-2\\-2\end{pmatrix} = 3\begin{pmatrix}1\\0\\2\end{pmatrix} - 2\begin{pmatrix}0\\1\\4\end{pmatrix},$$

so by distributivity,

$$Ax = A\left(3\begin{pmatrix}1\\0\\2\end{pmatrix} - 2\begin{pmatrix}0\\1\\4\end{pmatrix}\right) = 3A\begin{pmatrix}1\\0\\2\end{pmatrix} - 2A\begin{pmatrix}0\\1\\4\end{pmatrix} = 3\begin{pmatrix}7\\1\\2\\9\end{pmatrix} - 2\begin{pmatrix}1\\-4\\2\\3\end{pmatrix} = \begin{pmatrix}19\\11\\2\\21\end{pmatrix}.$$

17. For the following matrices A and B, compute AB, A^T, B^T, B^TA^T, and $(AB)^T$. Which of these matrices are equal and why? Why can't you compute A^TB^T?

$$A = \begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix} \qquad B = \begin{pmatrix}2 & 1 & -1\\4 & -1 & 2\end{pmatrix}.$$

Solution.

$$AB = \begin{pmatrix}10 & -1 & 3\\-8 & -1 & 0\end{pmatrix} \qquad A^T = \begin{pmatrix}1 & -2\\2 & -1\end{pmatrix}$$

$$B^T = \begin{pmatrix}2 & 4\\1 & -1\\-1 & 2\end{pmatrix} \qquad B^TA^T = \begin{pmatrix}10 & -8\\-1 & -1\\3 & 0\end{pmatrix} \qquad (AB)^T = \begin{pmatrix}10 & -8\\-1 & -1\\3 & 0\end{pmatrix}$$

The equality $(AB)^T = B^TA^T$ always holds, and we can't compute A^TB^T because we can't multiply a 2×2 matrix by a 3×2 matrix.

18. Recall that a matrix A is *symmetric* if $A^T = A$. Decide if each statement is true or false, and explain why.

a) If A and B are symmetric of the same size, then AB is symmetric.

b) If A is symmetric, then A^3 is symmetric.

c) If A is any matrix, then A^TA is symmetric.

Solution.

a) False.

b) True.

c) True.

19. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= 2 \\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Use row operations to eliminate x_1 from all but the first equation.

b) Use row operations to modify the system so that x_2 only appears in the first and second equations (and x_1 still only appears in the first).

c) Solve for x_3, then for x_2, then for x_1. What is the solution?

Solution.

a) We do $R_2 \mathrel{+}= 2R_1$ and $R_3 \mathrel{-}= 3R_1$ to obtain

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ x_2 + 7x_3 &= 4 \\ -x_2 - 10x_3 &= -1 \end{aligned}$$

b) We do $R_3 \mathrel{+}= R_2$ to obtain

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ x_2 + 7x_3 &= 4 \\ -3x_3 &= 3 \end{aligned}$$

c) $x_3 = -1 \qquad x_2 = 11 \qquad x_1 = 24$

20. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries of the table.

System of Equations	Matrix Equation	Augmented Matrix
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 + 4x_3 &= 2 \end{aligned}$		
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	
		$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

Solution.

System of Equations	Matrix Equation	Augmented Matrix
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 \quad\quad + 4x_3 &= 2 \end{aligned}$	$\begin{pmatrix} 3 & 2 & 4 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$	$\left(\begin{array}{ccc\|c} 3 & 2 & 4 & 9 \\ -1 & 0 & 4 & 2 \end{array}\right)$
$\begin{aligned} 3x_1 - 5x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \\ -x_1 + x_2 &= 2 \end{aligned}$	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	$\left(\begin{array}{cc\|c} 3 & -5 & 1 \\ 2 & 4 & 1 \\ -1 & 1 & 2 \end{array}\right)$
$\begin{aligned} x_1 \quad\quad + x_3 + x_4 &= 2 \\ 3x_2 - x_3 - 2x_4 &= 4 \\ x_1 - 3x_2 - 4x_3 - 3x_4 &= 2 \\ 6x_1 + 5x_2 - x_3 - 8x_4 &= 1 \end{aligned}$	$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & -1 & -2 \\ 1 & -3 & -4 & -3 \\ 6 & 5 & -1 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 1 \end{pmatrix}$	$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

21. Which of the following matrices are not in row echelon form? Why not?

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

Solution.

These are not in REF:

$$\begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

22. Find values of a and b such that the following system has **a)** zero, **b)** exactly one, and **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - \ y &= b \end{aligned}$$

Solution.

a) $a = -2$ and b any value except 2.

b) $a \neq -2$ and any value for b.

c) $(a, b) = (-2, 2)$.

23. Give examples of matrices A in *row echelon form* for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?

Solution.

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

There is no square matrix satisfying **b)**.