Math 218D-1: Homework #2

Answer Key

1. Which of the following matrices are not in reduced row echelon form? Why not?

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&3&4\end{pmatrix}\qquad\begin{pmatrix}3&0&1&0\\1&0&2&3\\0&0&0&4\end{pmatrix}\qquad\begin{pmatrix}1&0&4&0\\0&1&3&0\\0&0&0&1\end{pmatrix}\qquad\begin{pmatrix}1&3&4&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\qquad\begin{pmatrix}1&2&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad\begin{pmatrix}1&0&0&2\\0&1&0&1\\0&0&1&9\end{pmatrix}$$

Solution.

These matrices are not in RREF:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&3&4\end{pmatrix}\qquad\begin{pmatrix}3&0&1&0\\1&0&2&3\\0&0&0&4\end{pmatrix}$$

$$\begin{pmatrix}1&3&4&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\qquad\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\qquad\begin{pmatrix}1&2&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

2. Describe all possible nonzero 2×2 matrices in RREF.

Solution.

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad\begin{pmatrix}1&*\\0&0\end{pmatrix}\qquad\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

3. Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use Rabinoff's Reliable Row Reducer.

$$\textbf{a)}\begin{pmatrix}1&1&0\\1&2&1\\0&1&2\end{pmatrix}\qquad\textbf{b)}\left(\begin{array}{ccc|c}1&1&0&1\\1&1&1&1\\0&1&2&2\end{array}\right)\qquad\textbf{c)}\begin{pmatrix}1&2&0\\1&2&0\\0&1&2\end{pmatrix}$$

$$\textbf{d)}\begin{pmatrix}1&2&3&4\\4&5&6&7\\6&7&8&9\end{pmatrix}\qquad\textbf{e)}\left(\begin{array}{ccc|c}1&3&5&7\\3&5&7&9\\5&7&9&1\end{array}\right)\qquad\textbf{f)}\left(\begin{array}{ccccc|c}0&3&-6&6&4&-5\\3&-7&8&-5&8&9\\3&-9&12&-9&6&15\end{array}\right)$$

Solution.

Note that the row echelon form is not unique, but the RREF is unique. Pivots are in red.

a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right)$

c) $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

e) $\left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

f)

$$\left(\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{8}{3} \end{array}\right)$$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

4. Determine *how many* solutions each system of equations has. (Do not find the solutions.) [**Hint:** use Problem 3.]

$$\textbf{a)} \begin{cases} x_1 + x_2 \phantom{{}+ x_3} = 1 \\ x_1 + x_2 + x_3 = 1 \\ \phantom{x_1 +{}} x_2 + 2x_3 = 2 \end{cases} \qquad \textbf{b)} \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 9x_3 = 1 \end{cases}$$

$$\textbf{c)} \begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

Solution.

In each case we have found the pivots of the associated augmented matrix in Problem 3.

a) One solution: every column except the augmented column is a pivot column.

b) Zero solutions: the last column is a pivot column.

c) Infinitely many solutions: the third, fourth, and augmented columns have no pivot.

5. Use Gaussian elimination and back-substitution or Jordan substitution to solve

$$\textbf{a)}\begin{cases} x_1 + x_2 = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ x_2 + 2x_3 = 3 \end{cases} \qquad \textbf{b)}\begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 8x_3 = 1. \end{cases}$$

What happens if you replace 8 by 9 in (b)?

Solution.

a) $(x_1, x_2, x_3) = (2, -1, 2)$

b) $(x_1, x_2, x_3) = (8, -17, 10)$. The system

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 7 \\ 3x_1 + 5x_2 + 7x_3 &= 9 \\ 5x_1 + 7x_2 + 9x_3 &= 1 \end{aligned}$$

is inconsistent.

6. The parabola $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,9)$, $(-1,6)$. Find the coefficients a, b, c.

Solution.

$y = 2x^2 - x + 3$

7. Use the formula for the 2×2 inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

$$\textbf{a)}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \qquad \textbf{c)}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Solution.

a) $-\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

b) $-\frac{1}{2}\begin{pmatrix} 4 & -7 \\ -2 & 3 \end{pmatrix}$

c) Not invertible: $1 \cdot 4 - 2 \cdot 2 = 0$

8. Compute the inverses of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why.

$$\textbf{a)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 4 \\ -3 & 1 & 4 \end{pmatrix} \qquad \textbf{c)}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\textbf{d)}\begin{pmatrix} 6 & -4 & -7 & -1 \\ 7 & 0 & 1 & 3 \\ -1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

Solution.

a) $\begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

b) $\dfrac{1}{2}\begin{pmatrix} 16 & 2 & 6 \\ 20 & 2 & 8 \\ 7 & 1 & 3 \end{pmatrix}$

c) Not invertible: the RREF is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

d) $\dfrac{1}{2}\begin{pmatrix} 2 & -2 & 4 & 4 \\ 2 & -1 & 5 & 0 \\ 1 & -2 & 2 & 5 \\ -5 & 6 & -10 & -11 \end{pmatrix}$

9. Consider the linear system

$$\begin{aligned} x_1 + x_2 \phantom{{}+ x_3} &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the Problem 8 to solve for x_1, x_2, x_3 in terms of b_1, b_2, b_3. Do *not* use Gauss–Jordan elimination!

Solution.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3b_1 - 2b_2 + b_3 \\ -2b_1 + 2b_2 - b_3 \\ b_1 - b_2 + b_3 \end{pmatrix}$$

10. Suppose that

$$A\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

What is A^{-1}?

Solution.

$$A^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \\ 4 & 2 & 3 \end{pmatrix}.$$

(Multiply both sides of each equation by A^{-1}.)

11. Decide if each statement is true or false, and explain why.

a) If A and B are invertible $n \times n$ matrices, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.

b) If A is invertible then so is A^{10}.

c) An $n \times n$ matrix with n pivots is invertible.

d) An invertible $n \times n$ matrix has n pivots.

Solution.

a) False: $(AB)^{-1} = B^{-1}A^{-1}$.

b) True.

c) True.

d) True.

12. Consider a system of 3 equations in 4 variables. Write the elementary matrices that accomplish the following row operations:

a) $R_2 \mathrel{+}= 2R_1$ **b)** $R_1 \mathrel{-}= \frac{1}{2}R_3$ **c)** $R_3 \mathrel{\times}= 2$
d) $R_3 \mathrel{\div}= 2$ **e)** $R_1 \longleftrightarrow R_3$ **f)** $R_1 \longleftrightarrow R_2$

Solution.

a) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

e) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

f) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

13. Consider a system of 3 equations in 4 variables. Write the row operations that the following elementary matrices perform on that system:

$$\textbf{a)}\begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix}\qquad \textbf{b)}\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}\qquad \textbf{c)}\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}$$
$$\textbf{d)}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\qquad \textbf{e)}\begin{pmatrix}\frac{1}{4}&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Solution.

a) $R_1 \mathrel{+}= 3R_2$

b) $R_2 \mathrel{\times}= 3$

c) $R_3 \mathrel{-}= 3R_2$

d) $R_2 \longleftrightarrow R_3$

e) $R_1 \mathrel{\div}= 4$

14. For each elementary matrix in Problem 13, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \xrightarrow{\text{row op}} R_2 \mathrel{+}= R_1 \xrightarrow{\text{undo}} R_2 \mathrel{-}= R_1 \xrightarrow{\text{matrix}} \begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}$$
$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Solution.

a)

$$\begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix} \xrightarrow{\text{row op}} R_1 \mathrel{+}= 3R_2 \xrightarrow{\text{undo}} R_1 \mathrel{-}= 3R_2$$
$$\xrightarrow{\text{matrix}} \begin{pmatrix}1&-3&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix}^{-1}$$

b)

$$\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix} \overset{\text{row op}}{\rightsquigarrow} R_2 \times= 3 \overset{\text{undo}}{\rightsquigarrow} R_2 \div= 3$$

$$\overset{\text{matrix}}{\rightsquigarrow} \begin{pmatrix}1&0&0\\0&\frac{1}{3}&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}^{-1}$$

c)

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix} \overset{\text{row op}}{\rightsquigarrow} R_3 -= 3R_2 \overset{\text{undo}}{\rightsquigarrow} R_3 += 3R_2$$

$$\overset{\text{matrix}}{\rightsquigarrow} \begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}^{-1}$$

d)

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \overset{\text{row op}}{\rightsquigarrow} R_2 \longleftrightarrow R_3 \overset{\text{undo}}{\rightsquigarrow} R_2 \longleftrightarrow R_3$$

$$\overset{\text{matrix}}{\rightsquigarrow} \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}^{-1}$$

e)

$$\begin{pmatrix}\frac{1}{4}&0&0\\0&1&0\\0&0&1\end{pmatrix} \overset{\text{row op}}{\rightsquigarrow} R_1 \div= 4 \overset{\text{undo}}{\rightsquigarrow} R_1 \times= 4$$

$$\overset{\text{matrix}}{\rightsquigarrow} \begin{pmatrix}4&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}\frac{1}{4}&0&0\\0&1&0\\0&0&1\end{pmatrix}^{-1}$$

15. Consider the matrix

$$A = \begin{pmatrix}1&2&0\\1&3&2\\1&3&1\end{pmatrix}.$$

a) Explain how to reduce A to a matrix U in REF using three row replacements.

b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i:

$$U = \underline{\hspace{2cm}}A.$$

c) Fill in the blank with a product involving the E_i^{-1}:

$$A = \_\_\_\_\_\_\_\_ U$$

d) Evaluate that product to produce a lower-triangular matrix L with ones on the diagonal such that $A = LU$.

When multiplying elementary matrices, just use row operations!

e) Explain how to reduce U to the 3×3 identity matrix using three more row operations E_4, E_5, E_6.

f) Fill in the blank with a product involving the E_i:

$$A^{-1} = \_\_\_\_\_\_\_\_\_\_\_\_.$$

Solution.

a) $R_2 \mathrel{-}= R_1, \quad R_3 \mathrel{-}= R_1, \quad R_3 \mathrel{-}= R_2$

b) $U = E_3 E_2 E_1 A$

c) $A = E_1^{-1} E_2^{-1} E_3^{-1} U$

d) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

e) $R_3 \times= -1, \quad R_2 \mathrel{-}= 2R_3, \quad R_1 \mathrel{-}= 2R_2$

f) $A^{-1} = E_6 E_5 E_4 E_3 E_2 E_1$

16. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}.$$

a) Perform Gaussian elimination on A without using any row swaps. Write the REF matrix U you obtained.

b) Write the elementary matrices E_1, E_2, E_3 for the row operations you did in (a), with E_1 corresponding to the first row operation.

c) Compute the matrix $L = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$. [**Hint:** Don't multiply matrices! Recall that left-multiplication by E_i^{-1} "un-does" the ith row operation.]

d) Verify that L is lower-unitriangular and that $A = LU$.

Solution.

a) I performed the following row operations: $R_2 \mathrel{+}= R_1$, $R_3 \mathrel{+}= 3R_1$, and $R_3 \mathrel{+}= 2R_2$. I obtained the REF matrix

$$U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

b) The corresponding elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

c) The product is

$$L = (E_3E_2E_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix}.$$

d) We see that L is lower-unitriangular, and

$$\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

17. Solve the following matrix equations by forward- and back-substitution, using the provided LU decomposition. Check your answers by evaluating Ax.

a)
$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

b)
$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

c)
$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

Solution.

a)

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} y = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix} \implies y = \begin{pmatrix} 14 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix} x = \begin{pmatrix} 14 \\ 2 \\ 2 \end{pmatrix} \implies x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} y = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix} \implies y = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix} \implies x = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

c)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} y = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix} \implies y = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix} \implies x = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

18. Compute the $A = LU$ decomposition of the following matrices using the 2-column method. Check your answers by multiplying LU.

a) $\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$

Solution.

a) $L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

b) $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 4 & 1 \end{pmatrix}$ $U = \begin{pmatrix} 3 & 0 & 2 & -1 \\ 0 & -1 & 5 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$

c) $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 2 & -3 & 2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & -2 & -1 & 5 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

19. Solve the following matrix equations by forward- and back-substitution, using the provided $PA = LU$ decomposition. Check your answers by evaluating Ax.

a)
$$\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b)
$$\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} x = \begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Solution.

a)
$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix} y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix} = \begin{pmatrix} -14 \\ -59 \\ 54 \end{pmatrix} \implies y = \begin{pmatrix} -14 \\ -3 \\ -5 \end{pmatrix}$$
$$\begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix} x = \begin{pmatrix} -14 \\ -3 \\ -5 \end{pmatrix} \implies x = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

b)
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix} y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \\ 12 \end{pmatrix} \implies y = \begin{pmatrix} 4 \\ 4 \\ -5 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \\ -5 \\ 3 \end{pmatrix} \implies x = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

20. Compute a $PA = LU$ decomposition for each of the following matrices, using the 3-column method and performing *maximal partial pivoting*. Check your answers by multiplying PA and LU.

$$\textbf{a)}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$$

Solution.

a) There are two answers, depending on which row you choose for the first pivot. Choosing the second row:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

and choosing the third row:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{pmatrix}.$$

b) There are two answers, depending on which row you choose for the second pivot. Choosing the third row:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & -\frac{1}{4} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 4 & -9 \\ 0 & 0 & 0 & -\frac{1}{4} \end{pmatrix}$$

Choosing the fourth row:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & \frac{1}{4} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -\frac{1}{4} \end{pmatrix}$$

21. Recall that a *permutation matrix* is a product of elementary matrices for row swaps.

a) If P is the $n \times n$ elementary matrix for a row swap, explain why $P^{-1} = P = P^T$.

b) If P is any permutation matrix, show that $P^{-1} = P^T$. [**Hint:** write $P = P_1 P_2 \cdots P_r$ for elementary row swaps P_i.] Is $P = P^T$ for a general permutation matrix?

Solution.

a) We have $P^{-1} = P$ because $PP = I_n$ (swapping the same two rows twice cancels out). If P corresponds to $R_i \longleftrightarrow R_j$ then the (i,j) and (j,i) entries of P are both equal to 1, the (i,i) and (j,j) entries are zero, and otherwise P is equal to the identity matrix. Hence the same is true of P^T.

b) If $P = P_1 P_2 \cdots P_r$ then

$$P^{-1} = (P_1 P_2 \cdots P_r)^{-1} = P_r^{-1} \cdots P_2^{-1} P_1^{-1} = P_r \cdots P_2 P_1$$

and

$$P^T = (P_1 P_2 \cdots P_r)^T = P_r^T \cdots P_2^T P_1^T = P_r \cdots P_2 P_1.$$