Math 218D-1: Homework #3

Answer Key

1. Express each system of linear equations as a vector equation. For example,

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 - x_2 &= 4 \end{aligned} \quad \rightsquigarrow \quad x_1\begin{pmatrix}1\\-1\end{pmatrix} + x_2\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}.$$

a) $\begin{cases} 3x_1 + 2x_2 + 4x_3 = 9 \\ -x_1 + 4x_3 = 2 \end{cases}$ **b)** $\begin{pmatrix}3 & -5\\2 & 4\\-1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$

c) $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

Solution.

a) $x_1\begin{pmatrix}3\\-1\end{pmatrix} + x_2\begin{pmatrix}2\\0\end{pmatrix} + x_3\begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}9\\2\end{pmatrix}$

b) $x_1\begin{pmatrix}3\\2\\-1\end{pmatrix} + x_2\begin{pmatrix}-5\\4\\1\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$

c) $x_1\begin{pmatrix}1\\0\\1\\6\end{pmatrix} + x_2\begin{pmatrix}0\\3\\-3\\5\end{pmatrix} + x_3\begin{pmatrix}1\\-1\\-4\\-1\end{pmatrix} + x_4\begin{pmatrix}1\\-2\\-3\\-8\end{pmatrix} = \begin{pmatrix}2\\4\\2\\1\end{pmatrix}$

2. For each matrix A and vector b, decide if the system $Ax = b$ is consistent. If so, find the parametric form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix}1 & -1\\2 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \rightsquigarrow \quad x_1 = x_2 + 1.$$

Also answer the following questions: Which variables are free? How many solutions does the system have?

a) $A = \begin{pmatrix}2 & 1 & 1 & 4\\4 & 2 & 1 & 7\end{pmatrix} \qquad b = \begin{pmatrix}1\\1\end{pmatrix}$

b) $A = \begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix} \qquad b = \begin{pmatrix}3\\2\\49\end{pmatrix}$

c) $A = \begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix} \qquad b = \begin{pmatrix}3\\2\\48\end{pmatrix}$

d) $$A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

e) $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

Solution.

a) The free variables are x_2 and x_4, and there are infinitely many solutions:

$$\begin{aligned} x_1 &= -\tfrac{1}{2}x_2 - \tfrac{3}{2}x_4 \\ x_3 &= \qquad\;\; - \;\; x_4 + 1. \end{aligned}$$

b) The free variable is x_3, and there are infinitely many solutions:

$$\begin{aligned} x_1 &= -\tfrac{5}{2}x_3 + \tfrac{19}{2} \\ x_2 &= \;\;\; 3x_3 - \;\; 8. \end{aligned}$$

c) The system is inconsistent.

d) The free variables are x_2 and x_4, and there are infinitely many solutions:

$$\begin{aligned} x_1 &= -2x_2 + 7x_4 - 22 \\ x_3 &= \qquad\quad - 2x_4 + \;\; 8 \\ x_5 &= \qquad 0. \end{aligned}$$

e) There are no free variables. There is one solution

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 2 \\ x_3 &= 1. \end{aligned}$$

3. For each matrix A and vector b in Problem 2, find the parametric *vector* form of the general solution of $Ax = b$ (if the system is consistent). For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the dimension of the solution set?

Solution.

a) This is a plane in $\mathbf{R}^4$ (dimension 2):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

b) This is a line in $\mathbf{R}^3$ (dimension 1):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{19}{2} \\ -8 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix}$$

c) This is empty.

d) This is a plane in $\mathbf{R}^5$ (dimension 2):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -22 \\ 0 \\ 8 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

e) The "parametric" vector form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

The dimension is 0.

4. **a)** Is $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$?
If so, what are the weights?

b) Find a vector that is *not* a linear combination of the columns of the matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}.$$

[**Hint:** for both parts, compare Problem 2. This does not require any additional computation.]

Solution.

a) Setting $x_3 = 0$ in Problem 2(b), we get

$$\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix} = \frac{19}{2} \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} - 8 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}.$$

b) By Problem 2(c), one example is $(3, 2, 48)$.

5. The equation $x + 2y = z$ determines a plane in $\mathbf{R}^3$. (This is an *implicit equation* for the plane).

a) What is the coefficient matrix A for this system?

b) Which are the free variables?

c) Write the parametric form of the solutions of $x + 2y = z$. This expresses the points on the plane in terms of two *parameters*.

d) Do the same for the plane defined by $2y = z$. What is different?

Solution.

a) $\begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$

b) y and z

c) $x = -2y + z$

d) The coefficient matrix is $\begin{pmatrix} 0 & 2 & -1 \end{pmatrix}$, the free variables are x and z, and the parametric equation is $y = \frac{1}{2}z$. Here x is a free variable, but it does not appear in the parametric equation!

6. The equations

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \end{aligned}$$

determine a line $\mathbf{R}^3$. (These are *implicit equations* for the line). Write the line in parameterized form: that is, find three linear functions $f_1(t), f_2(t), f_3(t)$ in one variable such that all points on the line have the form $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ for a unique value of t. (Use the free variable as the parameter t.)

Solution.

The parametric form of the solution set of the given equations is

$$\begin{aligned} x &= \tfrac{1}{3} - \tfrac{1}{3}z \\ y &= -\tfrac{1}{3} - \tfrac{2}{3}z. \end{aligned}$$

Hence we can take $f_1(t) = \frac{1}{3} - \frac{1}{3}t$, $f_2(t) = -\frac{1}{3} - \frac{2}{3}t$, and $f_3(t) = t$.

7. Describe and compare (geometrically) the solution sets of the following systems:

$$\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ 4x_1 + 2x_2 + x_3 = 0 \end{cases} \qquad \begin{cases} 2x_1 + x_2 + x_3 = 1 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases}$$

This is much easier if you write the solutions in parametric vector form.

Solution.

They are *parallel* lines in $\mathbf{R}^3$.

8. Suppose that A is a 3×3 matrix and b is a vector such that the solution set of $Ax = b$ is a line in $\mathbf{R}^3$. How many pivots does A have?

Solution.

There is one free variable, hence **two** pivots.

9. Let R be a row echelon form of a matrix A. Explain why the following quantities are all equal to the rank of A:

a) The number of pivots of A.

b) The number of nonzero rows of R.

c) The number of columns of A minus the number of free variables.

Solution.

a) This is the definition.

b) Each nonzero row of R contains one pivot.

c) The free variables correspond to the columns without a pivot.

10. Decide if each statement is true or false, and explain why.

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An $m \times n$ matrix has at most m pivots.

d) A wide matrix (more columns than rows) must have a free variable.

e) If A is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.

Solution.

a) False.

b) True.

c) True.

d) True.

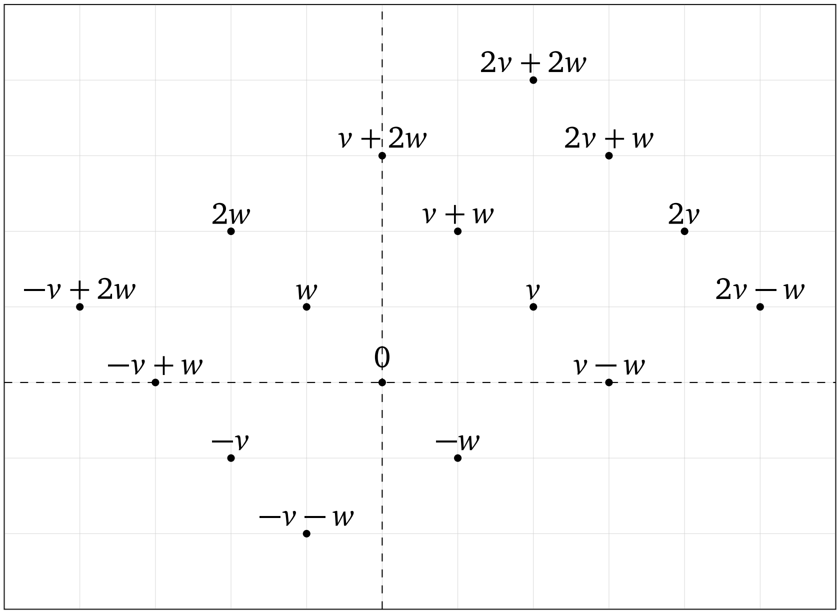
e) False.

11. Consider the vectors

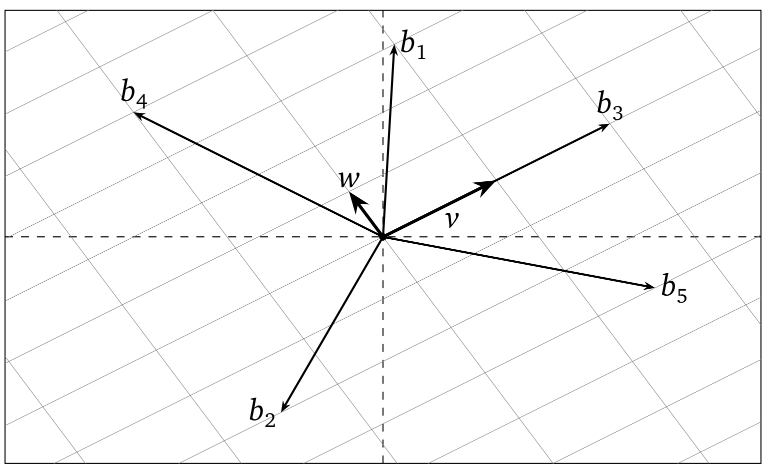
$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Draw the 16 linear combinations $cv + dw$ ($c, d = -1, 0, 1, 2$) as *points* in the xy-plane.

Solution.



12. Certain vectors v, w in $\mathbf{R}^2$ are drawn below. Express each of b_1, b_2, b_3, b_4, b_5 as a linear combination of v, w. *Do not try to guess the coordinates of v and w!* This is a question about the geometry of linear combinations.



Solution.

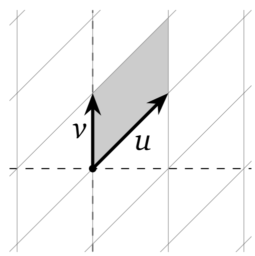
$$b_1 = v + 3w$$
$$b_2 = -\frac{3}{2}v - 2w$$
$$b_3 = 2v$$
$$b_4 = -v + 4w$$
$$b_5 = \frac{3}{2}v - 3w$$

13. Consider the vectors

$$u = \begin{pmatrix}1\\1\end{pmatrix}, \quad v = \begin{pmatrix}0\\1\end{pmatrix}.$$

Draw a picture of all of the linear combinations $au + bv$ for real numbers a, b satisfying $0 \le a \le 1$ and $0 \le b \le 1$. (This will be a shaded region in the xy-plane.)

Solution.



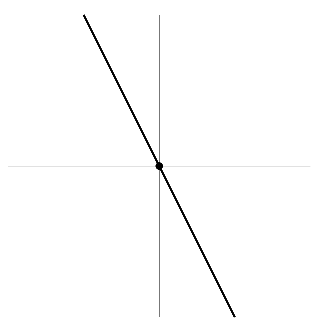
14. Draw a picture of all vectors $b \in \mathbf{R}^2$ for which the equation

$$\begin{pmatrix}1 & 2\\-2 & -4\end{pmatrix} x = b$$

is consistent. [**Hint:** the answer is a span!]

Solution.

This is equivalent to drawing all linear combinations of the columns $\binom{1}{-2}$ and $\binom{2}{-4}$. The columns are multiples of each other, so this is the line through the origin with slope -2:



15. For each matrix A and vector b, and express the solution set in the form

$$p + \operatorname{Span}\{???\}$$

for some vector p. For instance,

$$\begin{pmatrix}1 & -1\\2 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix}1\\0\end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

[**Hint:** You found the parametric vector form in Problem 3.]

a) $$A = \begin{pmatrix}2 & 1 & 1 & 4\\4 & 2 & 1 & 7\end{pmatrix} \qquad b = \begin{pmatrix}1\\1\end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

c) $$A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

d) $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

Solution.

a) $$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

b) $$\begin{pmatrix} \frac{19}{2} \\ -8 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix} \right\}$$

c) $$\begin{pmatrix} -22 \\ 0 \\ 8 \\ 0 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

d) $$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mathrm{Span}\{\}$$

16. For each matrix A in Problem 15, write the solution set of $Ax = 0$ as a span. Does there exist a nontrivial solution?

[**Hint:** this problem requires no additional computation.]

Solution.

a) $$\mathrm{Span}\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}; \text{ yes}$$

b) $$\mathrm{Span}\left\{ \begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix} \right\}; \text{ yes}$$

c) $$\text{Span}\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}7\\0\\-2\\1\\0\end{pmatrix}\right\}; \text{ yes}$$

d) Span{}; no

17. When is the following system consistent?

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= b_1 \\ -4x_1 - 5x_2 + 5x_3 &= b_2 \\ 6x_1 + x_2 + 12x_3 &= b_3 \end{aligned}$$

Your answer should be a single linear equation in b_1, b_2, b_3. [**Hint:** perform Gaussian elimination.]

Explain the relationship between this equation and

$$\text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}.$$

Solution.

The system is consistent when

$$-13b_1 - 5b_2 + b_3 = 0.$$

This is the implicit equation for the span of the columns of the associated coefficient matrix.

18. Let A be a 3×4 matrix whose columns span the plane $x + y + z = 0$.

a) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *consistent*.

b) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *inconsistent*.

[**Hint:** this problem requires no computations at all.]

Solution.

a) Any vector whose coordinates sum to zero.

b) Any vector whose coordinates do not sum to zero.

19. Suppose that $Ax = b$ is consistent. Explain in geometric terms why $Ax = b$ has a unique solution precisely when $Ax = 0$ has only the trivial solution.

Solution.

The solution set of $Ax = b$ is a translate of the solution set of $Ax = 0$.

20. Give geometric descriptions of the following spans (line, plane, ...).

a) $\text{Span}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **c)** $\text{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\-6\end{pmatrix}\right\}$

d) $\text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ **e)** $\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$

[**Hint:** for **d)**, compare Problem 17.]

Solution.

a) A line.

b) The yz-plane.

c) A line.

d) A plane.

e) All of $\mathbf{R}^3$.

21. **a)** List five nonzero vectors contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$.

b) Is $\begin{pmatrix}0\\3\\6\end{pmatrix}$ contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$?

If so, express $\begin{pmatrix}0\\3\\6\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}$.

c) Show that $\begin{pmatrix}7\\8\\9\end{pmatrix}$ is contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$.

d) Describe $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ geometrically.

e) Find a vector not contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$.

Solution.

a) For instance,

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} + 2\begin{pmatrix}4\\5\\6\end{pmatrix}.$$

b) Yes, $\begin{pmatrix}0\\3\\6\end{pmatrix} = 2\begin{pmatrix}1\\2\\3\end{pmatrix} + 3\begin{pmatrix}4\\5\\6\end{pmatrix} - 2\begin{pmatrix}7\\8\\9\end{pmatrix}$.

c) $\begin{pmatrix}7\\8\\9\end{pmatrix} = -\begin{pmatrix}1\\2\\3\end{pmatrix} + 2\begin{pmatrix}4\\5\\6\end{pmatrix}$.

d) It is a plane in $\mathbf{R}^3$.

e) Any vector not in this plane will work. For instance, $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

22. Decide if each statement is true or false, and explain why.

a) A vector b is a linear combination of the columns of A if and only if $Ax = b$ has a solution.

b) There is a matrix A such that $Ax = \binom{2}{2}$ has infinitely many solutions and $Ax = \binom{2}{-2}$ has exactly one solution.

c) The zero vector is contained in every span.

d) The matrix equation $Ax = 0$ can be consistent or inconsistent, depending on what A is.

e) If the zero vector is a solution of a system of equations, then the system is homogeneous.

f) If $Ax = b$ has a unique solution, then A has a pivot in every column.

g) If $Ax = b$ is consistent, then the solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

h) It is possible for $Ax = b$ to have exactly 13 solutions.

Solution.

a) True.

b) False.

c) True.

d) False: 0 is always a solution.

e) True.

f) True.

g) True.

h) False.