Math 218D-1: Homework #4

Answer Key

1. Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

$$\textbf{a)}\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

$$\textbf{c)}\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad \textbf{d)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** You already did all of the work in HW3#16.]

Solution.

a) The null space is a plane:

$$\operatorname{Nul}\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{pmatrix}\right\} = \operatorname{Col}\begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

b) The null space is a line:

$$\operatorname{Nul}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix}\right\} = \operatorname{Col}\begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix}.$$

c) The null space is a plane:

$$\operatorname{Nul}\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}\right\} = \operatorname{Col}\begin{pmatrix} -2 & 7 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

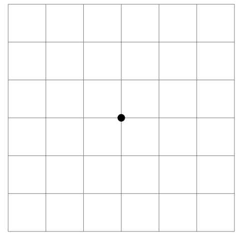
d) The null space is a point:

$$\operatorname{Nul}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \operatorname{Span}\{\} = \operatorname{Col}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

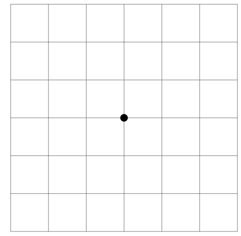
2. Draw pictures of the null space and the column space of the following matrices. Be precise!

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

Nul(A)

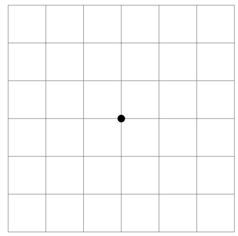


Col(A)

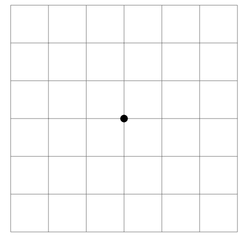


b) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

Nul(A)

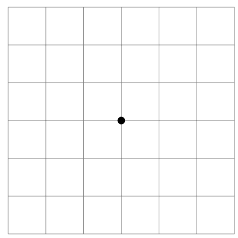


Col(A)

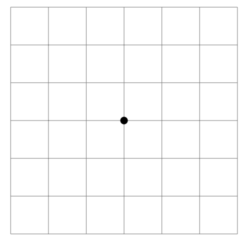


c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

Nul(A)



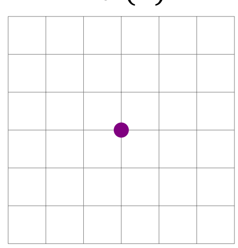
Col(A)



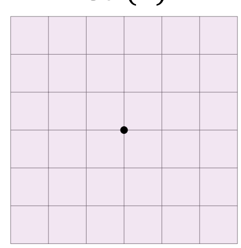
Solution.

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

Nul(A)

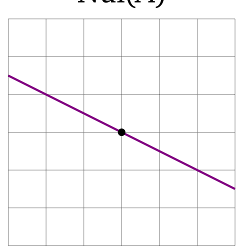


Col(A)

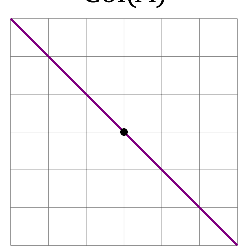


b) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

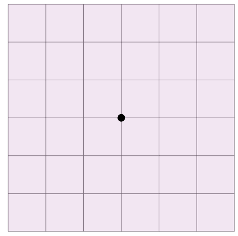
Nul(A)



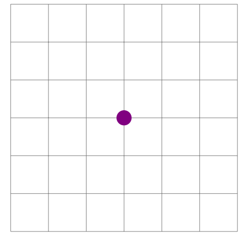
Col(A)



Nul(A)



Col(A)

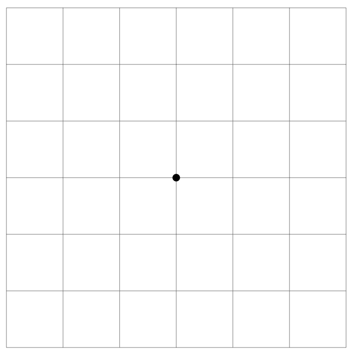


c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

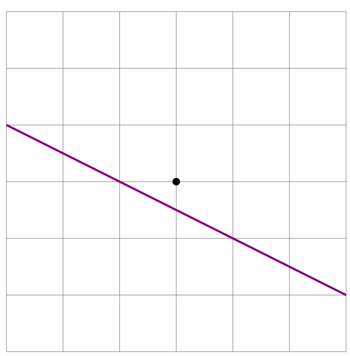
3. Draw the solution set of the matrix equation

$$\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Explain the relationship with your answer for Problem 2(b).



Solution.



This line is parallel to the one in Problem 2(b).

4. Let A be a matrix such that $\mathrm{Nul}(A) = \mathrm{Span}\{(1,1,-1,-1), (1,-1,1,-1)\}$. What is the rank of A, and why?

Solution.

The rank is 2, which is the number of columns (4) minus the number of free variables (2).

5. Give examples of subsets V of $\mathbf{R}^2$ such that:

a) V is closed under addition and contains 0, but is not closed under scalar multiplication.

b) V is is closed under scalar multiplication and contains 0, but is not closed under addition.

c) V is closed under addition and scalar multiplication, but does not contain 0.

Therefore, none of these conditions is redundant.

Solution.

a) There are many answers; one is the first quadrant.

b) There are many answers; one is the union of the x-axis and the y-axis.

c) The empty set is the only answer.

6. Which of the following subsets of $\mathbf{R}^3$ are subspaces? If it is not a subspace, find a counterexample to one of the subspace properties. If it is, express it as the column space or null space of some matrix.

a) The plane $\{(x, y, x) \colon x, y \in \mathbf{R}\}$.

b) The plane $\{(x, y, 1) \colon x, y \in \mathbf{R}\}$.

c) The set consisting of all vectors (x, y, z) such that $xy = 0$.

d) The set consisting of all vectors (x, y, z) such that $x \leq y$.

e) The span of $(1, 2, 3)$ and $(2, 1, -3)$.

f) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

g) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 1. \end{cases}$

Solution.

a) This is a subspace; it is equal to $\mathrm{Col}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b) This is not a subspace: it does not contain 0.

c) This is not a subspace: it is not closed under addition.

d) This is not a subspace: it is not closed under scalar multiplication.

e) Any span is a subspace; this is $\mathrm{Col}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & -3 \end{pmatrix}$.

f) This is $\operatorname{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \end{pmatrix}$, and any null space is a subspace.

g) This is not a subspace: it does not contain 0.

7. Give a geometric description of the following column spaces (line, plane, ...).

$$\textbf{a)}\ \operatorname{Col}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \textbf{b)}\ \operatorname{Col}\begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix} \qquad \textbf{c)}\ \operatorname{Col}\begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & -6 \end{pmatrix}$$

$$\textbf{d)}\ \operatorname{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad \textbf{e)}\ \operatorname{Col}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** Compare HW3#20.]

Solution.

a) This is a line.

b) This is the yz-plane.

c) This is a line.

d) This is a plane.

e) This is all of $\mathbf{R}^3$.

8. Find a nonzero 2×2 matrix such that $A^2 = 0$.

Solution.

There are many answers; one is $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

9. **a)** Explain why $\operatorname{Col}(AB)$ is contained in $\operatorname{Col}(A)$.

b) Give an example where $\operatorname{Col}(AB) \neq \operatorname{Col}(A)$.
[**Hint:** Take $A = B$ to be the matrix from Problem 8.]

Solution.

a) According to the column rule for matrix multiplication, the columns of AB are linear combinations of the columns of A.

b) There are many examples; one is $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

10. **a)** Explain why $\operatorname{Nul}(AB)$ contains $\operatorname{Nul}(B)$.

b) Give an example where $\operatorname{Nul}(AB) \neq \operatorname{Nul}(B)$.
[**Hint:** Take $A = B$ to be the matrix from Problem 8.]

Solution.

a) If $Bx = 0$ then $ABx = A0 = 0$.

b) There are many examples; one is $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

11. Decide if each statement is true or false, and explain why.

a) The column space of an $m \times n$ matrix with m pivots is a subspace of $\mathbf{R}^m$.

b) The null space of an $m \times n$ matrix with n pivots is equal to $\mathbf{R}^n$.

c) If $\mathrm{Col}(A) = \{0\}$, then A is the zero matrix.

d) The column space of $2A$ equals the column space of A.

e) The null space of $A + B$ contains the null space of A.

f) If U is an echelon form of A, then $\mathrm{Nul}(U) = \mathrm{Nul}(A)$.

g) If U is an echelon form of A, then $\mathrm{Col}(U) = \mathrm{Col}(A)$.

Solution.

a) True.

b) False: it is $\{0\}$.

c) True.

d) True.

e) False: take $A = 0$ and $B = I_n$.

f) True: the null space is a solution set.

g) False.

12. **a)** Give an example of a 3×3 matrix A such that $\mathrm{Col}(A)$ contains $(1, 2, 3)$ and $(1, 0, -1)$, but $\mathrm{Col}(A)$ is not all of $\mathbf{R}^3$. What is the rank of A?

b) Give an example of a 3×3 matrix A, with no zero entries, such that $\mathrm{Col}(A)$ is the line through $(1, 1, 1)$. What is the rank of A?

Solution.

a) There are many answers, although they all have rank 2. One way to proceed is to take $(1, 2, 3)$ and $(1, 0, -1)$ as the first two columns; then the third must be a linear combination of the first two:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & -1 & 2 \end{pmatrix}.$$

b) All columns must be multiples of $(1,1,1)$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The rank is 1.

13. For each matrix A, verify that $\operatorname{rank}(A) = 1$, and find vectors u, v such that $A = uv^T$. (In general, a matrix has rank 1 if and only if it is equal to a column vector times a row vector.)

$$\textbf{a) } A = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix} \qquad \textbf{b) } A = \begin{pmatrix} 2 & 1 & -1 & 4 \\ -2 & -1 & 1 & -4 \\ 2 & 1 & -1 & 4 \end{pmatrix}$$

Solution.

a)

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

b)

$$A = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 & 4 \end{pmatrix}$$

14. Find a basis for the null space of each matrix.

$$\textbf{a)} \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

$$\textbf{c)} \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad \textbf{d)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** You already did all of the work in Problem 1.]

Solution.

a)

$$\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{3}{2} \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

b)

$$\left\{ \begin{pmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{pmatrix} \right\}$$

c)
$$\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}7\\0\\-2\\1\\0\end{pmatrix}\right\}$$

d)
$$\{\}$$

15. Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear relation among them.

a) $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ **b)** $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$ **c)** $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$

d) $\left\{\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix}, \begin{pmatrix}2\\-4\\2\\-6\end{pmatrix}, \begin{pmatrix}3\\-5\\2\\-7\end{pmatrix}, \begin{pmatrix}-1\\4\\-3\\7\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\6\end{pmatrix}\right\}$ **e)** $\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$

Which sets do you know are linearly dependent without doing any work?

Solution.

a) These are linearly dependent:
$$\begin{pmatrix}1\\2\\3\end{pmatrix} - 2\begin{pmatrix}4\\5\\6\end{pmatrix} + \begin{pmatrix}7\\8\\9\end{pmatrix} = 0.$$

b) These are linearly dependent:
$$0\begin{pmatrix}1\\2\end{pmatrix} + 0\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = 0.$$

You don't have to do any work to verify linear dependence: the set contains the zero vector.

c) These are linearly independent.

d) These are linearly dependent:
$$-2\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix} + \begin{pmatrix}2\\-4\\2\\-6\end{pmatrix} + 0\begin{pmatrix}3\\-5\\2\\-7\end{pmatrix} + 0\begin{pmatrix}-1\\4\\-3\\7\end{pmatrix} + 0\begin{pmatrix}1\\1\\-1\\6\end{pmatrix} = 0.$$

(There are many other linear relations.)

You don't have to do any work to verify linear dependence: these are six vectors in $\mathbf{R}^4$.

e) These are linearly independent.

16. **a)** For each set in Problem 15, find a basis for the span of the vectors.

b) For each set in Problem 15, find a *different* basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for **a)**.

c) What is the dimension of each of these spans?

Solution.

a) These are the bases one would compute by using pivot columns:

$$\textbf{a)}\ \left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix}\right\}\qquad \textbf{b)}\ \left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right\}\qquad \textbf{c)}\ \left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

$$\textbf{d)}\ \left\{\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix},\begin{pmatrix}3\\-5\\2\\-7\end{pmatrix},\begin{pmatrix}1\\1\\-1\\6\end{pmatrix}\right\}\qquad \textbf{e)}\ \left\{\begin{pmatrix}0\\1\\3\end{pmatrix},\begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$$

b) There are many answers here. An easy way to solve this problem is by observing that $\operatorname{Span}\{v,w\} = \operatorname{Span}\{v+w, v-w\}$ for any vectors v and w, and using Problem 18 for **d)**.

$$\textbf{a)}\ \left\{\begin{pmatrix}5\\7\\9\end{pmatrix},\begin{pmatrix}3\\3\\3\end{pmatrix}\right\}\qquad \textbf{b)}\ \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\qquad \textbf{c)}\ \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$\textbf{d)}\ \left\{\begin{pmatrix}4\\-4\\1\\-1\end{pmatrix},\begin{pmatrix}2\\-1\\0\\3\end{pmatrix},\begin{pmatrix}4\\-7\\3\\-10\end{pmatrix}\right\}\qquad \textbf{e)}\ \left\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

c) The dimensions are: **a)** 2 **b)** 2 **c)** 3 **d)**3 **e)**2.

17. Consider the vectors

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix},\begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$$

of Problem 15(a).

a) Find two different ways to express $(5,7,9)$ as a linear combination of these vectors.

b) How many ways can you express $(5,7,9)$ as a linear combination of the first two vectors? (This does not require elimination to answer.)

Solution.

a)
$$\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4\\5\\6\end{pmatrix}+0\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}5\\7\\9\end{pmatrix}=2\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}4\\5\\6\end{pmatrix}+\begin{pmatrix}7\\8\\9\end{pmatrix}$$

b) There is only one way because the set

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$$

is linearly independent.

18. Let $\{w_1, w_2, w_3\}$ be a basis for a subspace V, and set

$$v_1 = w_2 + w_3 \qquad v_2 = w_1 + w_3 \qquad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for V.
[**Hint:** You have to check spanning and linear independence by hand.]

Solution.

First we check linear independence: suppose that

$$\begin{aligned} 0 = x_1v_1 + x_2v_2 + x_3v_3 &= x_1(w_2 + w_3) + x_2(w_1 + w_3) + x_3(w_1 + w_2) \\ &= w_1(x_2 + x_3) + w_2(x_1 + x_3) + w_3(x_1 + x_2). \end{aligned}$$

Since $\{w_1, w_2, w_3\}$ is linearly independent, this means

$$\begin{aligned} x_2 + x_3 &= 0 \\ x_1 \qquad + x_3 &= 0 \\ x_1 + x_2 \qquad &= 0 \end{aligned} \quad\Longrightarrow\quad \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0.$$

One checks that the matrix in the above equation has full column rank, which implies $x_1 = x_2 = x_3 = 0$.

Now we check that $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{w_1, w_2, w_3\}$. Since each v_i is a linear combination of the w_i, the first span is contained in the second. Hence we need to show that any linear combination of the w_i is also a linear combination of the v_i: i.e., for any b_1, b_2, b_3, there exist x_1, x_2, x_3 such that

$$b_1w_1 + b_2w_2 + b_3w_3 = x_1v_1 + x_2v_2 + x_3v_3.$$

Substituting for v_1, v_2, v_3, this reads

$$\begin{aligned} b_1w_1 + b_2w_2 + b_3w_3 &= x_1v_1 + x_2v_2 + x_3v_3 \\ &= x_1(w_2 + w_3) + x_2(w_1 + w_3) + x_3(w_1 + w_2) \\ &= w_1(x_2 + x_3) + w_2(x_1 + x_3) + w_3(x_1 + x_2). \end{aligned}$$

Comparing coefficients of the w_i, we want to solve

$$\begin{aligned} x_2 + x_3 &= b_1 \\ x_1 \qquad + x_3 &= b_2 \\ x_1 + x_2 \qquad &= b_3 \end{aligned} \quad\Longrightarrow\quad \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}.$$

Again, the matrix in the above equation has full row rank, so a solution always exists.

19. Certain vectors v_1, v_2, v_3, v_4 span a 3-dimensional subspace of $\mathbf{R}^5$. They satisfy the linear relation

$$2v_1 + 0v_2 - v_3 + v_4 = 0.$$

a) Describe *all* linear relations among v_1, v_2, v_3, v_4.
[**Hint:** what is the rank of the matrix with columns v_1, v_2, v_3, v_4?]

b) Which vector(s) is/are *not* in the span of the others? How do you know for sure?

Solution.

a) The matrix A with columns v_1, v_2, v_3, v_4 has rank 3 because its column space has dimension 3. Hence it has three pivots, so it has one free variable; thus its null space has dimension 1, so it must be spanned by $(2, 0, -3, 1)$. It follows that all linear relations are multiples of $2v_1 + 0v_2 - v_3 + v_4 = 0$.

b) The vector v_2 is not in $\mathrm{Span}\{v_1, v_3, v_4\}$: otherwise there would be a linear relation of the form $av_1 + bv_2 + cv_3 + dv_4 = 0$ with $b \neq 0$, which is not a multiple of the linear relation given in the problem.

20. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

Which sets of columns form a basis for the column space? (I.e., do the first and third columns form a basis? what about the second and third? etc.)

Solution.

The column space is a plane, and no two columns are collinear, so any pair of columns is a basis for the null space.

21. Find bases for the following subspaces.

a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

Solution.

a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ **b)** $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ **c)** $\left\{ \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \right\}$ **e)** $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ **f)** $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

(There are generally many choices of basis.)

22. Decide if each statement is true or false, and explain why.

a) If $v_1, v_2, \ldots, v_n$ are linearly independent vectors, then $\mathrm{Span}\{v_1, v_2, \ldots, v_n\}$ has dimension n.

b) If the matrix equation $Ax = 0$ has the trivial solution, then the columns of A are linearly independent.

c) If $\mathrm{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \mathrm{Span}\{v_1, v_2\}$.

d) If v_3 is not a linear combination of v_1 and v_2, then $\{v_1, v_2, v_3\}$ is linearly independent.

e) If $\{v_1, v_2, v_3\}$ is linearly dependent, then so is $\{v_1, v_2, v_3, x\}$ for any vector x.

f) The set $\{0\}$ is linearly independent.

g) If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then so is $\{v_1, v_2, v_3\}$.

h) The columns of any 4×5 matrix are linearly dependent.

i) If $Ax = \binom{1}{2}$ has only one solution, then the columns of A are linearly independent.

j) If $\mathrm{Span}\{v_1, v_2, v_3\}$ has dimension 3, then $\{v_1, v_2, v_3\}$ is linearly independent.

Solution.

a) True.

b) False.

c) True.

d) False.

e) True.

f) False.

g) True.

h) True.

i) True.

j) True.