Math 218D-1: Homework #4

due Wednesday, September 25, at 11:59pm

1. Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

a) $\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

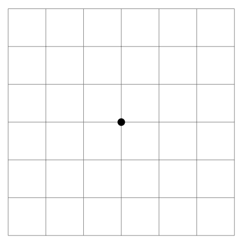
c) $\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

[**Hint:** You already did all of the work in HW3#16.]

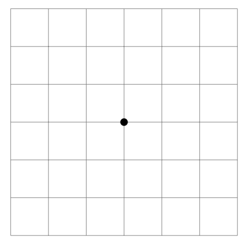
2. Draw pictures of the null space and the column space of the following matrices. Be precise!

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

Nul(A)

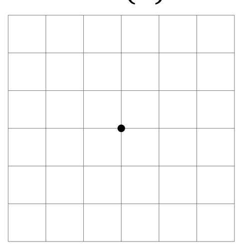


Col(A)

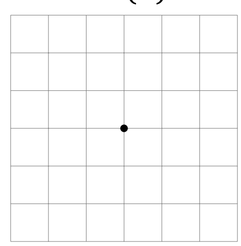


b) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

Nul(A)

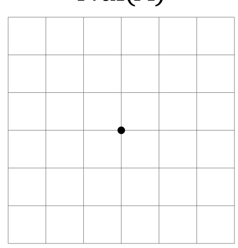


Col(A)

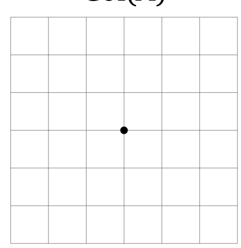


c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

Nul(A)



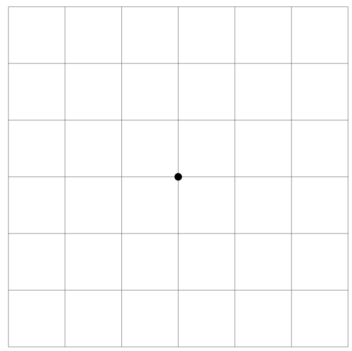
Col(A)



3. Draw the solution set of the matrix equation

$$\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Explain the relationship with your answer for Problem 2(b).



4. Let A be a matrix such that $\mathrm{Nul}(A) = \mathrm{Span}\{(1,1,-1,-1), (1,-1,1,-1)\}$. What is the rank of A, and why?

5. Give examples of subsets V of $\mathbf{R}^2$ such that:

a) V is closed under addition and contains 0, but is not closed under scalar multiplication.

b) V is is closed under scalar multiplication and contains 0, but is not closed under addition.

c) V is closed under addition and scalar multiplication, but does not contain 0.

Therefore, none of these conditions is redundant.

6. Which of the following subsets of $\mathbf{R}^3$ are subspaces? If it is not a subspace, find a counterexample to one of the subspace properties. If it is, express it as the column space or null space of some matrix.

a) The plane $\{(x,y,x) : x,y \in \mathbf{R}\}$.

b) The plane $\{(x,y,1) : x,y \in \mathbf{R}\}$.

c) The set consisting of all vectors (x,y,z) such that $xy = 0$.

d) The set consisting of all vectors (x,y,z) such that $x \leq y$.

e) The span of $(1,2,3)$ and $(2,1,-3)$.

f) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

g) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 1. \end{cases}$

7. Give a geometric description of the following column spaces (line, plane, . . .).

$$\textbf{a)}\ \mathrm{Col}\begin{pmatrix}2\\2\\1\end{pmatrix} \qquad \textbf{b)}\ \mathrm{Col}\begin{pmatrix}0&0\\1&-2\\3&1\end{pmatrix} \qquad \textbf{c)}\ \mathrm{Col}\begin{pmatrix}0&0\\1&-2\\3&-6\end{pmatrix}$$

$$\textbf{d)}\ \mathrm{Col}\begin{pmatrix}2&2&-1\\-4&-5&5\\6&1&12\end{pmatrix} \qquad \textbf{e)}\ \mathrm{Col}\begin{pmatrix}1&1&0\\1&2&1\\0&1&2\end{pmatrix}$$

[**Hint:** Compare HW3#20.]

8. Find a nonzero 2×2 matrix such that $A^2 = 0$.

9. **a)** Explain why $\mathrm{Col}(AB)$ is contained in $\mathrm{Col}(A)$.

b) Give an example where $\mathrm{Col}(AB) \neq \mathrm{Col}(A)$.
[**Hint:** Take $A = B$ to be the matrix from Problem 8.]

10. **a)** Explain why $\mathrm{Nul}(AB)$ contains $\mathrm{Nul}(B)$.

b) Give an example where $\mathrm{Nul}(AB) \neq \mathrm{Nul}(B)$.
[**Hint:** Take $A = B$ to be the matrix from Problem 8.]

11. Decide if each statement is true or false, and explain why.

a) The column space of an $m \times n$ matrix with m pivots is a subspace of $\mathbf{R}^m$.

b) The null space of an $m \times n$ matrix with n pivots is equal to $\mathbf{R}^n$.

c) If $\mathrm{Col}(A) = \{0\}$, then A is the zero matrix.

d) The column space of $2A$ equals the column space of A.

e) The null space of $A + B$ contains the null space of A.

f) If U is an echelon form of A, then $\mathrm{Nul}(U) = \mathrm{Nul}(A)$.

g) If U is an echelon form of A, then $\mathrm{Col}(U) = \mathrm{Col}(A)$.

12. **a)** Give an example of a 3×3 matrix A such that $\mathrm{Col}(A)$ contains $(1,2,3)$ and $(1,0,-1)$, but $\mathrm{Col}(A)$ is not all of $\mathbf{R}^3$. What is the rank of A?

b) Give an example of a 3×3 matrix A, with no zero entries, such that $\mathrm{Col}(A)$ is the line through $(1,1,1)$. What is the rank of A?

13. For each matrix A, verify that $\mathrm{rank}(A) = 1$, and find vectors u, v such that $A = uv^T$. (In general, a matrix has rank 1 if and only if it is equal to a column vector times a row vector.)

$$\textbf{a)}\ A = \begin{pmatrix}4&5&6\\8&10&12\\12&15&18\end{pmatrix} \qquad \textbf{b)}\ A = \begin{pmatrix}2&1&-1&4\\-2&-1&1&-4\\2&1&-1&4\end{pmatrix}$$

14. Find a basis for the null space of each matrix.

$$\text{a)}\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad \text{b)}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

$$\text{c)}\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad \text{d)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** You already did all of the work in Problem 1.]

15. Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear relation among them.

$$\text{a)}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix},\begin{pmatrix}7\\8\\9\end{pmatrix}\right\} \qquad \text{b)}\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right\} \qquad \text{c)}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

$$\text{d)}\left\{\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix},\begin{pmatrix}2\\-4\\2\\-6\end{pmatrix},\begin{pmatrix}3\\-5\\2\\-7\end{pmatrix},\begin{pmatrix}-1\\4\\-3\\7\end{pmatrix},\begin{pmatrix}1\\1\\-1\\6\end{pmatrix}\right\} \qquad \text{e)}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix},\begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$$

Which sets do you know are linearly dependent without doing any work?

16.

a) For each set in Problem 15, find a basis for the span of the vectors.

b) For each set in Problem 15, find a *different* basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for **a)**.

c) What is the dimension of each of these spans?

17. Consider the vectors

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix},\begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$$

of Problem 15(a).

a) Find two different ways to express $(5,7,9)$ as a linear combination of these vectors.

b) How many ways can you express $(5,7,9)$ as a linear combination of the first two vectors? (This does not require elimination to answer.)

18. Let $\{w_1, w_2, w_3\}$ be a basis for a subspace V, and set

$$v_1 = w_2 + w_3 \qquad v_2 = w_1 + w_3 \qquad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for V.

[**Hint:** You have to check spanning and linear independence by hand.]

19. Certain vectors v_1, v_2, v_3, v_4 span a 3-dimensional subspace of $\mathbf{R}^5$. They satisfy the linear relation

$$2v_1 + 0v_2 - v_3 + v_4 = 0.$$

a) Describe *all* linear relations among v_1, v_2, v_3, v_4.
[**Hint:** what is the rank of the matrix with columns v_1, v_2, v_3, v_4?]

b) Which vector(s) is/are *not* in the span of the others? How do you know for sure?

20. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

Which sets of columns form a basis for the column space? (I.e., do the first and third columns form a basis? what about the second and third? etc.)

21. Find bases for the following subspaces.

a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

22. Decide if each statement is true or false, and explain why.

a) If $v_1, v_2, \ldots, v_n$ are linearly independent vectors, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ has dimension n.

b) If the matrix equation $Ax = 0$ has the trivial solution, then the columns of A are linearly independent.

c) If $\text{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \text{Span}\{v_1, v_2\}$.

d) If v_3 is not a linear combination of v_1 and v_2, then $\{v_1, v_2, v_3\}$ is linearly independent.

e) If $\{v_1, v_2, v_3\}$ is linearly dependent, then so is $\{v_1, v_2, v_3, x\}$ for any vector x.

f) The set $\{0\}$ is linearly independent.

g) If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then so is $\{v_1, v_2, v_3\}$.

h) The columns of any 4×5 matrix are linearly dependent.

i) If $Ax = \binom{1}{2}$ has only one solution, then the columns of A are linearly independent.

j) If $\text{Span}\{v_1, v_2, v_3\}$ has dimension 3, then $\{v_1, v_2, v_3\}$ is linearly independent.