Math 218D-1: Homework #5

due Wednesday, October 2, at 11:59pm

1. Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that:

(1) $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A)$ is the number of columns of A.

(2) $\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A^T)$ is the number of rows of A.

(3) $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$.

[**Hint:** Augment with the identity matrix so you only have to do Gauss–Jordan elimination once.]

$$\textbf{a)}\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad \textbf{c)}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\textbf{d)}\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad \textbf{e)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Feel free to use the Sage cell on the website! For instance, to solve **a)** you might input:

```
A = Matrix([[ 2, 1, 1, 4, 1, 0],
            [ 4, 2, 1, 7, 0, 1]])
pprint(A.rref(pivots=False))
```

2. Suppose that A is an invertible 4×4 matrix. Find bases for its four fundamental subspaces.

[**Hint:** No calculations are necessary.]

3.
a) Let A be a 9×4 matrix of rank 3. What are the dimensions of its four fundamental subspaces?

b) If the left null space of a 5×4 matrix A has dimension 3, what is the rank of A?

4. Find an example of a matrix with the required properties, or explain why no such matrix exists.

a) The column space contains $(1,2,3)$ and $(4,5,6)$, and the row space contains $(1,2)$ and $(2,3)$.

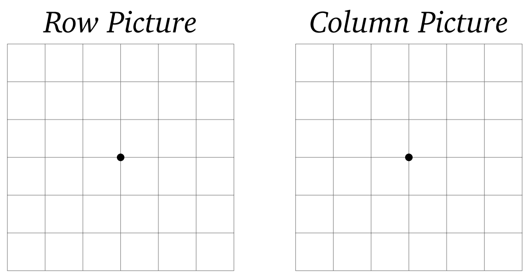
b) The column space has basis $\{(1,2,3)\}$, and the null space has basis $\{(3,2,1)\}$.

c) The dimension of the null space is one greater than the dimension of the left null space.

d) A 3×5 matrix whose row space equals its null space.

5. Draw the four fundamental subspaces of the following matrices, in grids like below.

$$\textbf{a)}\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$$



6. For the following matrix A, find the pivot positions of A and of A^T. Do they have the same pivots? Do they have the same rank?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{pmatrix}$$

7. Find a matrix A such that

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

What is the rank of A?

8. **a)** If $\mathrm{Col}(B)$ is contained in $\mathrm{Nul}(A)$, then $AB =$ __________.

b) Find a 2×2 matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$. What is the rank of such a matrix? [**Hint:** use HW4#8.]

9. **a)** Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$. [**Hint:** Compare HW4#9.]

b) Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$. [**Hint:** Take transposes in (a).]

10. Let A be a 3×3 matrix of rank 2. Explain why A^2 is not the zero matrix.
[**Hint:** Compare Problem 8.]

11. This problem explains why we only consider *square* matrices when we discuss invertibility.

a) Show that a tall matrix A (more rows than columns) does not have a right inverse, i.e., there is no matrix B such that $AB = I_m$.

b) Show that a wide matrix A (more columns than rows) does not have a left inverse, i.e., there is no matrix B such that $BA = I_n$.

[**Hint:** Use Problem 9.]

12. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "A has full column rank"?

a) $\mathrm{Nul}(A) = \{0\}$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \mathrm{Row}(A) = n$

e) The columns of A span $\mathbf{R}^m$

f) A^T has full column rank

13. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "A has full row rank"?

a) $\mathrm{Col}(A) = \mathbf{R}^m$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \mathrm{Nul}(A) = n - m$

e) The rows of A span $\mathbf{R}^n$

f) A^T has full column rank

14. Consider the following vectors:

$$u = \begin{pmatrix} -.6 \\ .8 \end{pmatrix} \quad v = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a) Compute the lengths $\|u\|, \|v\|$, and $\|w\|$.

b) Compute the lengths $\|2u\|, \|-v\|$, and $\|3w\|$.

c) Find the unit vectors in the directions of u, v, and w.

d) Check the Schwartz inequalities $|u \cdot v| \leq \|u\|\,\|v\|$ and $|v \cdot w| \leq \|v\|\,\|w\|$.

e) Find the angles between u and v and between v and w.

f) Find the distance from v to w.

g) Find unit vectors u', v', w' orthogonal to u, v, w, respectively.

15. If $\|v\| = 5$ and $\|w\| = 3$, what are the smallest and largest possible values of $\|v - w\|$? What are the smallest and largest possible values of $v \cdot w$? Justify your answer using the algebra of dot products.

16. **a)** If $v \cdot w < 0$, what does that say about the angle between v and w?

b) Find three vectors u, v, w in the xy-plane such that $u \cdot v < 0$, $u \cdot w < 0$, and $v \cdot w < 0$. (Draw a picture!)

17. Compute a basis for the orthogonal complement of each of the following spans.

a) $\text{Span}\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$ **c)** $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix},\begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$

d) $\text{Span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ **e)** $\text{Span}\{\} = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

f) $\text{Span}\left\{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\}$

18. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\text{Col}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **b)** $\text{Nul}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **c)** $\text{Row}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$

d) $\text{Nul}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ **e)** $\text{Span}\left\{\begin{pmatrix}3\\1\\3\end{pmatrix},\begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **f)** $\text{Col}\begin{pmatrix}1&2&0\\2&1&0\end{pmatrix}$

[**Hint:** solving **a)–d)** requires only one Gauss-Jordan elimination, and **f)** doesn't require any work.]

19. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\{(x,y,x) : x,y \in \mathbf{R}\}$.

b) $\{(x,y,z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix}0&6&8\\ \frac{1}{2}&0&0\\ 0&\frac{1}{2}&0\end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

g) The line $\{(t,-t,t) : t \in \mathbf{R}\}$.

[**Hint:** Compare HW4#21.]

20. Construct a matrix A with each of the following properties, or explain why no such matrix exists.

a) The column space contains $(0,2,1)$, and the null space contains $(1,-1,2)$ and $(-1,3,2)$.

b) The row space contains $(0,2,1)$, and the null space contains $(1,-1,2)$ and $(-1,3,2)$.

c) $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent, and $A^T \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 0$.

d) A 2×2 matrix A with no zero entries such that every row of A is orthogonal to every column.

e) The sum of the columns of A is $(0,0,0)$, and the sum of the rows of A is $(1,1,1)$.