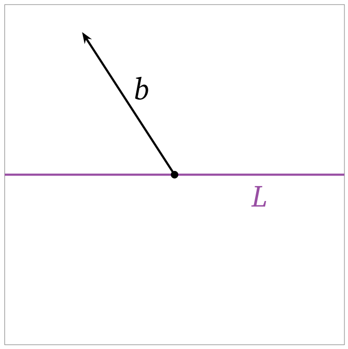
Math 218D-1: Homework #6

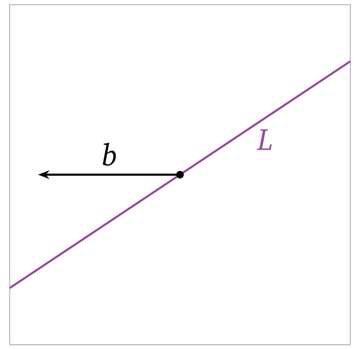
due Wednesday, October 9, at 11:59pm

1. In each case, *draw* the orthogonal projection b_V of b onto L *without doing any computations*.

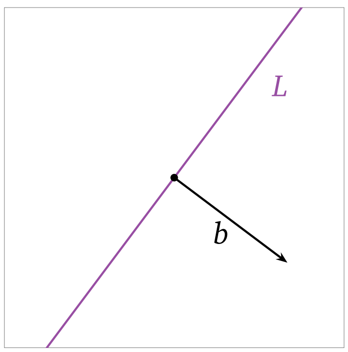
a)



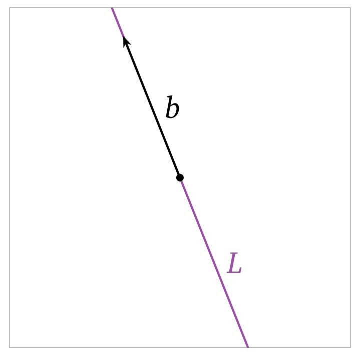
b)



c)



d)



2. For each subspace V and vector b, compute the orthogonal projection b_V of b onto V by solving a normal equation $A^TAx = A^Tb$, and find the distance from b to V.

a) $$V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

b) $$V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$$

c) $$V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$$

3. For each subspace V, compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1,2,-1)$ with respect to V. Use the formula for projection onto a line in **c)**.

$$\textbf{a) } V = \operatorname{Span}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}\right\} \qquad \textbf{b) } V = \operatorname{Nul}\begin{pmatrix}1 & 2 & 2\\0 & 2 & 0\end{pmatrix}$$

$$\textbf{c) } V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \qquad \textbf{d) } V = \mathbf{R}^3 \qquad \textbf{e) } V = \{0\}$$

[**Hint:** Only part **a)** requires elimination.]

4. Compute the orthogonal decomposition $(3,1,3) = b_V + b_{V^\perp}$ with respect to each subspace of V of HW5#18(a)–(e).

[**Hint:** Only parts **a)** and **c)** require any work, and even **c)** doesn't require work if you're clever enough. In fact, you can solve all five parts by computing two dot products.]

5. **a)** Let $v, w \in \mathbf{R}^n$. Show that

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

if $v \perp w$.

b) Let V be a subspace of $\mathbf{R}^n$, let $b \in \mathbf{R}^n$, and let $v \in V$. Use **a)** and the fact that $b - b_V \in V^\perp$ to show that

$$\|b - v\|^2 = \|b - b_V\|^2 + \|b_V - v\|^2.$$

Use this to prove that b_V really is the closest vector in V to b.

c) Let V be a subspace of $\mathbf{R}^n$ and let $b \in \mathbf{R}^n$. Use **a)** to show that $\|b_V\| \leq \|b\|$, with equality if and only if $b \in V$.

6. **a)** Find an implicit equation for the plane

$$\operatorname{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}.$$

[**Hint:** use HW5#18(a).]

b) Find implicit equations for the line $\{(t, -t, t) : t \in \mathbf{R}\}$.

[**Hint:** use HW5#19(g).]

7. Show that $A^T A = 0$ is only possible when $A = 0$.

8. Let Q be an $n \times n$ (*square*) matrix such that $Q^T Q = I_n$ (so $Q^T = Q^{-1}$).

a) Show that the columns of Q are unit vectors.

b) Show that the columns of Q are orthogonal to each other.

c) Show that the *rows* of Q are also orthogonal unit vectors.

d) Find all 2×2 matrices Q such that $Q^T Q = I_2$.

Such a matrix Q is called *orthogonal*.[1]

9. Construct a 3×3 matrix A, with no zero entries, whose columns are orthogonal to each other. Compute $A^T A$, and explain why this matrix is diagonal.

10. Explain why A has full column rank if and only if $A^T A$ is invertible.

11. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal complements.

b) If A is a 3×4 matrix, then $\mathrm{Col}(A)^\perp$ is a subspace of $\mathbf{R}^4$.

c) If A is any matrix, then $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A)$.

d) If A is any matrix, then $\mathrm{Row}(A) = \mathrm{Row}(A^T A)$.

e) If every vector in a subspace V is orthogonal to every vector in another subspace W, then $V = W^\perp$.

f) If $x \in V$ and $x \in V^\perp$, then $x = 0$.

g) If x is in a subspace V, then the orthogonal projection of x onto V is x.

h) If x is in the orthogonal complement of a subspace V, then the orthogonal projection of x onto V is x.

12. For each column space V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

a) $V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ **b)** $V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix}$ **c)** $V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

13. For each subspace V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

a) $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\}$ **b)** $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

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[1] I am not responsible for this terminology.

14. For each vector v, compute the projection matrix onto $V = \operatorname{Span}\{v\}$ using the formula $P_V = vv^T / v \cdot v$.

$$\textbf{a) } v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \textbf{b) } v = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} \qquad \textbf{c) } v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (\text{in } \mathbf{R}^n)$$

15. Compute P_V and $P_{V^\perp}$ when V is the column space of an invertible $n \times n$ matrix.

16. For each subspace V, compute the projection matrix P_V.

a) $\{(x, y, x) \colon x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 \colon x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 \colon Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

g) The line $\{(t, -t, t) \colon t \in \mathbf{R}\}$.

[**Hint:** Compare HW4#21 and HW5#19. You can save a lot of work by sometimes computing $P_{V^\perp}$ and using $P_V = I_3 - P_{V^\perp}$.]

17. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 3 \end{pmatrix},$$

and let $V = \operatorname{Col}(A)$.

a) Compute P_V using the formula $P_V = A(A^T A)^{-1} A^T$.

b) Compute a basis $\{v_1, v_2\}$ for $V^\perp = \operatorname{Nul}(A^T)$.

c) Let B be the matrix with columns v_1, v_2, and compute $P_{V^\perp}$ using the formula $B(B^T B)^{-1} B^T$.

d) Verify that your answers to (a) and (c) sum to I_4.

(Factor out $ad - bc$ and use a computer to do the matrix multiplication! Your answers should be in fractions, not decimals.)

This illustrates the fact that once you've computed P_V, there's no need to compute $P_{V^\perp}$ separately. It's a lot of extra work!

18. Compute the matrices P_1, P_2 for orthogonal projection onto the lines through $a_1 = (-1,2,2)$ and $a_2 = (2,2,-1)$, respectively. Now compute P_1P_2, and explain why it is what it is.

19. Consider the plane V defined by the equation $x + 2y - z = 0$. Compute the matrix P_V for orthogonal projection onto V in two ways:

a) Find a basis for V, put your basis vectors into a matrix A, and use the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute the matrix for orthogonal projection $P_{V^\perp}$ onto the line $V^\perp$ using the formula $vv^T/v \cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.

[**Hint:** It doesn't take any work to find a basis for $V^\perp$.]

If V is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

20. Decide if each statement is true or false, and explain why. In each statement, V is a subspace of $\mathbf{R}^n$.

a) The rank of P_V is equal to $\dim(V)$.

b) $P_V P_{V^\perp} = 0$.

c) $P_V + P_{V^\perp} = 0$.

d) $\mathrm{Col}(P_V) = V$.

e) $\mathrm{Nul}(P_V) = V$.

f) $\mathrm{Row}(P_V) = \mathrm{Col}(P_V)$.

g) $\mathrm{Nul}(P_V)^\perp = \mathrm{Col}(P_V)$.