# Welcome to Math 218D-1!

## Introduction to Linear Algebra

What is Linear Algebra?

The study of (systems of) linear equations

Like: $y = 3x + 2 \rightsquigarrow -3x + y = 2$

(usually put variables on the left & constants on the right)

Or: $\begin{cases} x + y + z = 1 \\ y - z = -3 \end{cases}$ ; solve both equations at once

(arrange in columns to keep things tidy)

Linear means: equations that involve only sums of (number)·(variable) or (number)

Not: $xy + z = 1$ ← product of variables

$x + 3 = y^2$ ← power of a variable

$e^x = \cos(y)$ ← complicated functions

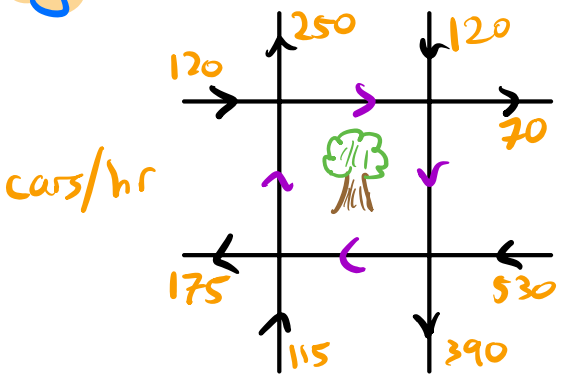
Why linear algebra?

- It's simple enough to understand very well & program computers to do quickly.

- It's powerful enough to solve a huge range of different problems.

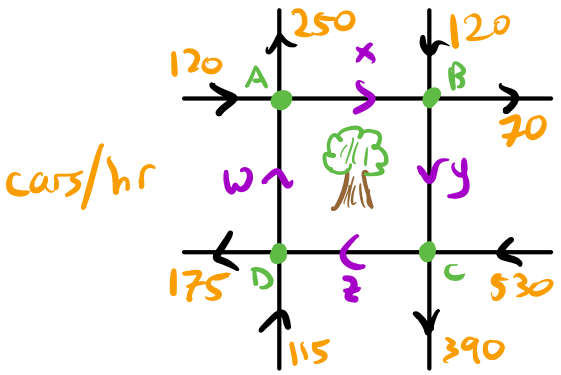
"example"

Eg: Here's a map of roads in the town square:




Question: How many cars/hr travel on the unlabeled roads?

Step 0: When you have an unknown quantity, give it a name!




Observation:

#cars entering each intersection = #cars leaving it

(notice nice columns)

A: $120 + w = 250 + x$

B: $120 + x = 70 + y$

C: $530 + y = 390 + z$

D: $115 + z = 175 + w$

$$\rightsquigarrow \begin{cases} -x + w = 130 \\ x - y = -50 \\ y - z = -140 \\ z - w = 60 \end{cases}$$

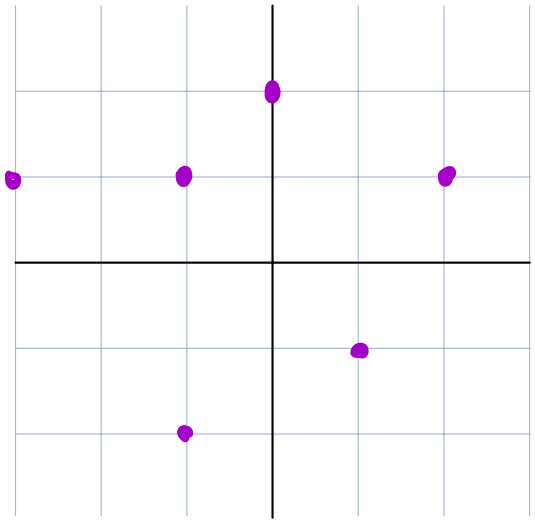
This is a system of 4 linear equations in 4 unknowns!

**Question:** You know a priori that there are infinitely many solutions. How?

**Question:** What must be true about the known quantities for a solution to exist?

Linear algebra is a set of tools for solving equations. It is your job to turn your question into a linear algebra problem (that a computer can solve) and interpret the answer.

**Eg:** An asteroid has been observed at coordinates:
$(0,2)$, $(2,1)$, $(1,-1)$, $(-1,-2)$, $(-3,1)$, $(-1,1)$



**Question:** What is the most likely orbit? Will the asteroid crash into the Earth?

**Fact:** The orbit is an ellipse.

**Equation for an ellipse:**

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

**Wait!** Isn't this a nonlinear equation? ...

For our points to lie on the ellipse, substitute the coordinates into $(x,y)$ ⇝ these should hold:

$x$ $y$
$\|$ $\|$

$(0,2)$: $0 + 4B + 0 + 0 + 2E + F = 0$

$(2,1)$: $4 + B + 2C + 2D + E + F = 0$

$(1,-1)$: $1 + B - C + D - E + F = 0$

$(-1,-2)$: $1 + 4B + 2C - D - 2E + F = 0$

$(-3,1)$: $9 + B - 3C + D - 3E + F = 0$

$(-1,1)$: $1 + B - C - D + E + F = 0$

This is a system of six linear equations in 5 variables.

"Note"

NB: The variables are the coefficients $B, C, D, E, F$. Remember, we're finding the equation of the ellipse.

NB: There is no solution — the points do not lie on an ellipse (perhaps due to measurement error).

Question: What is the best approximate solution?

⇝ "least squares" (week 8)

Answer: [demo]

**Historical note:** Gauss invented much of what you'll learn to (correctly) predict the orbit of the asteroid Ceres in 1801.

**Note on demos:** I created these to help give you a geometric understanding of linear algebra.

→ It took a lot of work.

→ Really, it was hard.

→ Why would I do that? I want you to have a geometric understanding.

**Upshot:** Play with the demos! Don't turn off your brain when we do geometry! You will be expected to draw pictures on exams!

**Eg:** In a population of rabbits,

(1) Half survive their first year ☹

(2) Half of those survive their second year.

(3) The maximum life span is 3 years.

(4) Each rabbit produces (on average) 0, 6, 8 offspring in years 0, 1, 2, respectively.

**Question:** How many rabbits will there be in 100 years?

**Step 0:** Give names to the unknowns.

$x_n$: # rabbits aged 0 in year $n$

$y_n$: # rabbits aged 1 in year $n$

$z_n$: # rabbits aged 2 in year $n$

**Rules:**

$$x_{2021} = 6y_{2020} + 8z_{2020}$$

$$y_{2021} = \frac{1}{2}x_{2020}$$

$$z_{2021} = \frac{1}{2}y_{2020}$$

A system of equations of this form is called a difference equation. We'll solve them using eigenvalues & diagonalization (week 10).

[demo] It looks like eventually,

- The population doubles each year
- The ratio of rabbits aged 0 : 1 : 2 is $\approx$ 16 : 4 : 1

**Comes from:** $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{bmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$ w/eigenvalue 2.

**Other examples:**

- Google PageRank lets you search the Web with a Markov chain — a special type of difference equation.

- Netflix knows what movies you'll like using the Singular Value Decomposition (weeks 14-15).

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# Geometry of Solutions

**Convention:** given a system of linear equations, put the constant term on the right of the $=$, and put the variables on the left, organized in columns.

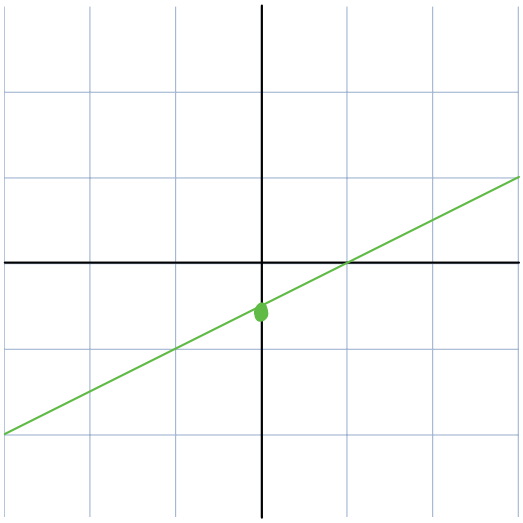
$$\begin{aligned} 120 + w &= 250 + x \\ 120 + x &= 70 + y \\ 530 + y &= 390 + z \\ 115 + z &= 175 + w \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} -x \qquad\quad + w &= 130 \\ x - y \qquad\quad &= -50 \\ y - z \qquad\quad &= -140 \\ z - w &= 60 \end{aligned}$$

**Def:** The solution set of a system of equations is the set of all values for the variables making all equations true simultaneously.

**Question:** What does the solution set of a system of linear equations look like?

One equation in 2 variables:

$x - 2y = 1 \rightsquigarrow y = \frac{1}{2}x - \frac{1}{2}$



One equation in 3 variables:

$x + y + z = 1 \rightsquigarrow z = 1 - x - y$

plane in xyz-space

[demo]

One equation in 4 variables: "3-plane in 4-space"

Note on dimensions: Students often want to say "the fourth dimension is time." Einstein used $\mathbb{R}^4$ (4-space) to model spacetime, but it models lots of other things too. (like traffic around the town square...)

2 equations in 2 variables:

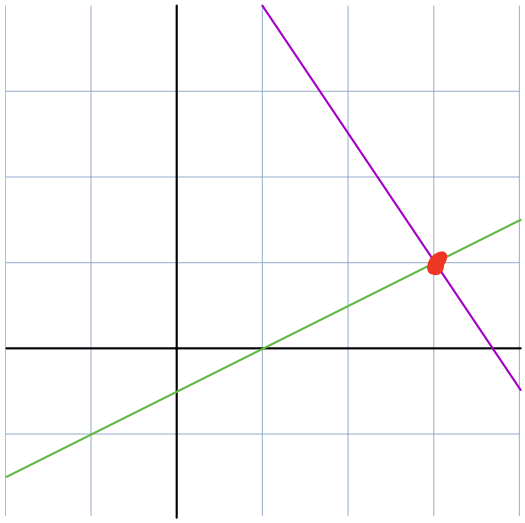
$x - 2y = 1$

$3x + 2y = 11$

Where are both true?

Intersection of 2 lines.

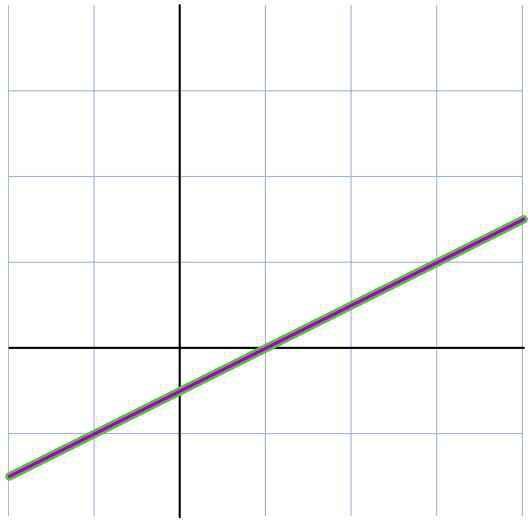
(answer: (3,1))



# What else can happen?

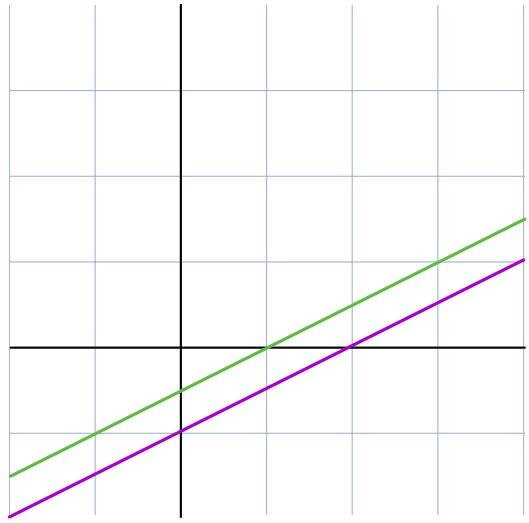
$$x - 2y = 1$$
$$3x - 6y = 3$$

Same line: $\infty$ solutions.



$$x - 2y = 1$$
$$3x - 6y = 6$$

Parallel lines: 0 solutions



## 2 equations in 3 variables:

$$x + y + z = 1$$
$$x \quad - z = 0$$

$\rightsquigarrow$ intersection of two planes in space [demo]

In this case, it's a line.

## 3 equations in 3 variables:

$$x + y + z = 1$$
$$x \quad - z = 0$$
$$y \quad = 0$$

$\rightsquigarrow$

$$x = 1/2$$
$$y = 0$$
$$z = 1/2$$

intersection of three planes in space: in this case it's a point

**Question:** How many "ways" can 3 planes in space intersect?

**Answer:** 8

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**Syllabus Stuff:** see the syllabus for details.

- Course materials, calendar, resources, links, etc. are on the course webpage:

https://services.math.duke.edu/~jdr/2425f-218/

- We will use Canvas for:
  - → Announcements
  - → Gradebook
  - → Gradescope
    - !! Please use the Gradescope tab on Canvas instead of going to gradescope.com.
  - → Ed Discussion: for asking questions (replaces Piazza).
    - !! Don't email us w/ math questions! Post it here instead – then everyone sees it & anyone can answer.
  - → WarpWire (see below)

**Textbook:**

- Strang, "Introduction to Linear Algebra", 5th ed.

We'll only follow this loosely. Also see

- Margalit-Rabinoff, "Interactive Linear Algebra" (on the course website).

**Quizzes:** a 10-minute small-group quiz will be held at the beginning of each discussion section. It's very basic – just tests if you've looked over your notes.

**Homework:** due Wednesday 11:59pm every week.

- Meant to be long and hard: you need practice to learn math, and practice takes time.
- I won't test you on any concept you didn't see on the homework.
- Scan & submit on Gradescope. Use a scanning app!
- Tag the pages on Gradescope with the problems on that page!

**Midterms:** 2 of them, during discussion slots.

**Final:** as scheduled by the registrar.

Help! • Come to office hours!
- Ask on Ed Discussion
- See course webpage.

Recorded Lecture:

Basics of vector & matrix algebra.
Watch before Tuesday (on Warp Wire)
HW#1 covers that material.