# The Basis Theorem

Recall from last time:

$$\boxed{\text{Basis of } \mathbb{R}^n \equiv \text{cols of an invertible } n\times n \text{ matrix}}$$

For an $n\times n$ matrix,

full col rank $\Longleftrightarrow$ invertible $\Longleftrightarrow$ full row rank

In terms of columns, $n$ vectors in $\mathbb{R}^n$

spans $\mathbb{R}^n$ $\Longleftrightarrow$ linearly independent

This is a special case of the basis theorem.

**Basis Theorem:** Let $V$ be a subspace of dim $d$

(1) If $d$ vectors span $V$ then they're a basis

(2) If $d$ vectors in $V$ are LI then they're a basis.

So if you have the correct number of vectors, you only need to check one of spans / LI.

**Eg:** • Two noncollinear vectors in a plane form a basis.

• Two vectors that span a plane form a basis.

This is how the Basis Thm makes our intuition precise.

# Geometry of Dot Products

We are now aiming to find the "best" approximate solution of $Ax=b$ when no actual solution exists.

Eg: find the best-fit ellipse through these points from the 1st lecture...

Q: How close can $Ax$ get to $b$?

$$\mathrm{Col}(A) = \{Ax : x \in \mathbb{R}^n\}$$

so this means: what is the closest vector $\hat{b}$ in $\mathrm{Col}(A)$ to $b$?

A: $b - \hat{b}$ is perpendicular to $\mathrm{Col}(A)$

[demo]

So we want to understand what vectors are perpendicular to a subspace.

We will study the geometric notion of "perpendicular" using the algebra of dot products.

Recall: $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \rightarrow v \cdot w = x_1 y_1 + \cdots + x_n y_n = v^T w$

$$\left( v^T w = (x_1 \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (x_1 y_1 + \cdots + x_n y_n) = \underset{1\times 1 \text{ matrix}}{(v \cdot w)} \right)$$

Dot products measure length & angles ← (eg. 90°)

→ geometric questions about length & angles become algebraic questions about dot products.

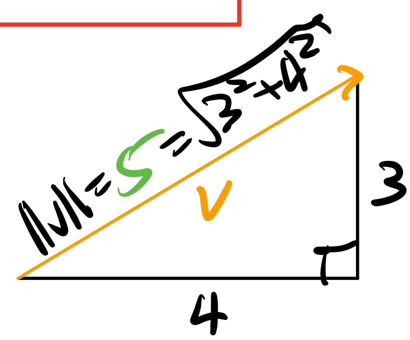
**Recall:** If $v = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ then

$$v \cdot v = x_1^2 + x_2^2 + \dots + x_n^2 \geq 0$$

**Def:** The length of $v$ is

$$\boxed{\|v\| = \sqrt{v \cdot v} \qquad \text{ie} \qquad \|v\|^2 = v \cdot v}$$

This makes sense by the Pythagorean theorem: $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$




**Sanity Check:** $c \in \mathbb{R}$ $v \in \mathbb{R}^n$

$$\|cv\| = \left\| c \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \right\| = \sqrt{(cx_1)^2 + \dots + (cx_n)^2}$$

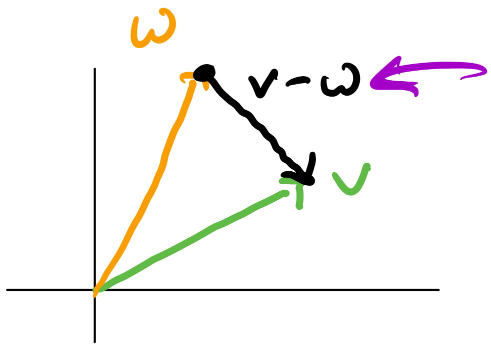
$$= |c| \cdot \sqrt{x_1^2 + \dots + x_n^2} = |c| \cdot \|v\| \quad \checkmark$$

$$\boxed{\|cv\| = |c| \cdot \|v\|}$$

**Eg:** $2v$ is twice as long as $v$.

So is $-2v$.

**Def:** The distance from v to w is $\|v-w\| = \|w-v\|$




length of v-w is distance from v to w

**Def:** A unit vector is a vector of length 1.

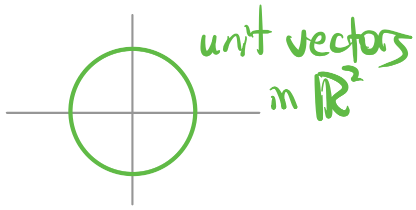
ie $\|v\| = 1$ ie. $\|v\|^2 = v \cdot v = 1$

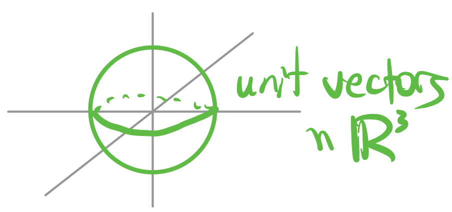
If $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then v is a unit vector

$\iff x_1^2 + \cdots + x_n^2 = 1$

$\iff$ v lies on the unit $(n-1)$-sphere

($n=2$: unit circle)



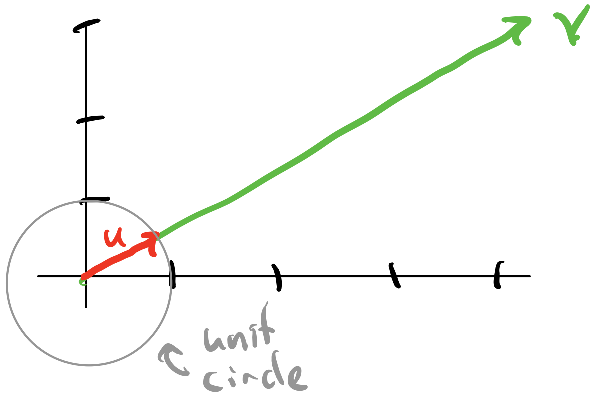

If $v \neq 0$, the unit vector in the direction of v is the vector

$$u = \frac{1}{\|v\|} \cdot v = \frac{v}{\|v\|} \quad \text{(scalar} \times \text{vector)}$$

**NB:** $\|u\| = \left| \frac{1}{\|v\|} \right| \cdot \|v\| = \frac{\|v\|}{\|v\|} = 1$ ✓

**Eg:** $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\quad \|v\| = \sqrt{3^2+4^2} = 5$

$$u = \frac{1}{\|v\|} v = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$$

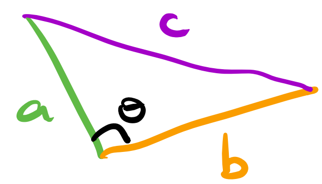



**NB:** all unit vectors in $\mathbb{R}^2$ are on the unit circle.

What about $v \cdot w$ for $v \neq w$?

**Law of Cosines:**

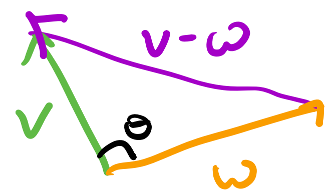
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$




**Vector Version:**

$$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$$

$$(a = \|v\| \quad b = \|w\| \quad c = \|v-w\|)$$




**Algebra:** "left hand side" LHS: $\|v-w\|^2 := (v-w)\cdot(v-w)$

$$\overset{\text{FOIL}}{=} v\cdot v + w\cdot w - 2v\cdot w$$

$$= \|v\|^2 + \|w\|^2 - 2v\cdot w$$

"right hand side" RHS $= \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$

cancel my

$$\boxed{v\cdot w = \|v\|\|w\|\cos\theta \quad \text{or} \quad \cos\theta = \frac{v\cdot w}{\|v\|\|w\|} \quad (\text{if } v,w \neq 0)}$$

**Def:** The angle from $v$ to $w$ $(v, w \neq 0)$ is

$$\theta := \cos^{-1}\left(\frac{v \cdot w}{\|v\|\|w\|}\right)$$

**NB:** $|\cos\theta| = \left|\frac{v\cdot w}{\|v\|\|w\|}\right| \in [0,1]$

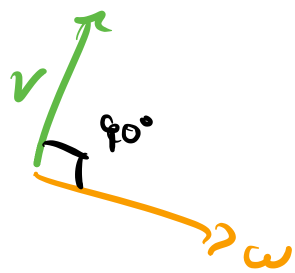
$$\Longrightarrow \quad |v\cdot w| \le \|v\|\cdot\|w\|$$

**Schwartz Inequality:** $|v\cdot w| \le \|v\|\cdot\|w\|$

**Def:** Vectors $v$ and $w$ are orthogonal or perpendicular, written $v \perp w$, if $\boxed{v\cdot w = 0}$

This says that either:

- $v=0$ or $w=0$ (or both), or
- $\cos(\theta)=0 \iff \theta = \pm 90^\circ$




**NB:** The zero vector is orthogonal to every vector:

$$0\cdot v = 0 \quad \text{for all } v$$

# Orthogonality

We want to know: "Which vectors are $\perp$ a subspace?"
Let's start with: "which vectors are $\perp$ some vector?"

Eg: Find all vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

We need to solve $v \cdot x = 0$

$$\Longleftrightarrow v^T x = 0$$

This is just $\mathrm{Nul}(v^T)$:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \rightsquigarrow x_1 + x_2 + x_3 = 0$$

$$\overset{PF}{\rightsquigarrow} \quad \begin{aligned} x_1 &= -x_2 - x_3 \\ x_2 &= \phantom{-}x_2 \\ x_3 &= \phantom{-x_2 -}x_3 \end{aligned}$$

$$\overset{PVF}{\rightsquigarrow} \quad x = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

[demo] $\rightsquigarrow \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ a plane

Check: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ ✓

Eg: Find all vectors orthogonal to $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ & $v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$

We need to solve $\begin{cases} v_1^T \cdot x = 0 \\ v_2^T \cdot x = 0 \end{cases} \rightsquigarrow \begin{matrix} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 \qquad = 0 \end{matrix}$

Equivalently, $\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix} \cdot x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \end{pmatrix} = 0$

So we want $\mathrm{Nul}\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{PF}} \begin{matrix} x_1 = -x_2 \\ x_2 = \phantom{-}x_2 \\ x_3 = \phantom{-}0 \end{matrix}$$

$$\xrightarrow{\text{PVF}} x = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$ a line

Check: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$ ✓

[demo]

NB: If $x \perp v_1$ and $x \perp v_2$ then

$$x \cdot (av_1 + bv_2) = a\, x\cdot v_1 + b\, x \cdot v_2 = a\cdot 0 + b \cdot 0 = 0$$

So $x$ is orthogonal to every vector in $\mathrm{Span}\{v_1, v_2\}$

[demo again]

More generally,

$$\left\{ v \in \mathbb{R}^n : \begin{array}{l} v \text{ is orthogonal} \\ \text{to every vector} \\ \text{in } \mathrm{Span}\{v_1, \dots, v_n\} \end{array} \right\} = \mathrm{Nul}\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}$$

This is awkward to say — let's give it a name.

Def: Let $V$ be a subspace of $\mathbb{R}^n$.

The orthogonal complement of $V$ is

$$V^\perp = \left\{ w \in \mathbb{R}^n : \begin{array}{l} w \text{ is orthogonal to} \\ \text{every vector in } V \end{array} \right\}$$

NB: Note the difference in notations:

- $V^\perp$ is the orthogonal complement of a subspc
- $A^T$ is the transpose of a matrix.

**NB:** If $x$ is in both $V$ and $V^\perp$ then $x$ is orthogonal to itself:

$$x \cdot x = 0 \implies x = 0, \text{ so } V \cap V^\perp = \{0\}$$

intersect

So we showed above:

$$\boxed{\operatorname{Span}\{v_1, \dots, v_n\}^\perp = \operatorname{Nul}\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}}$$

**Eg:** $V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \implies V^\perp = \operatorname{Nul}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$

**Eg:** $V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} \implies V^\perp = \operatorname{Nul}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix}$

**Eg:** $\{0\}^\perp = \mathbb{R}^n \qquad (\mathbb{R}^n)^\perp = \{0\}$

**Fact:** $V^\perp$ is also a subspace of $\mathbb{R}^n$.

Check:

(1) Let $x, y \in V^\perp$. So $x \cdot v = 0$ and $y \cdot v = 0$ for every $v \in V$. So $(x+y) \cdot v = x \cdot v + y \cdot v = 0 + 0$ for every $v \in V \rightsquigarrow x+y \in V^\perp$.

(2) Let $x \in V^\perp$, $c \in \mathbb{R}$. So $x \cdot v = 0$ for every $v \in V$.
So $(cx) \cdot v = c(x \cdot v) = c \cdot 0 = 0$ for every $v \in V$
$\rightsquigarrow cx \in V^\perp$.

(3) $0 \cdot v = 0$ for every $v \in V \rightsquigarrow 0 \in V^\perp$.

Or:

Every subspace is a span, and the orthogonal complement of a span is a null space (which is a subspace).

**Facts:** Let $V$ be a subspace of $\mathbb{R}^n$.

(1) $\dim(V) + \dim(V^\perp) = n$

(2) $(V^\perp)^\perp = V$

[demos]

**NB:** (2) says $V$ and $V^\perp$ are orthogonal complements of each other. Subspaces come in orthogonal complement pairs.

# Orthogonality of the Four Subspaces

**Recall:** If someone gives you a subspace, **Step 0** is to write it as a column space or a null space. So we want to understand $\mathrm{Col}(A)^\perp$ & $\mathrm{Nul}(A)^\perp$.

Let $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$. Then

$$\mathrm{Col}(A)^\perp = \mathrm{Span}\{v_1, \dots, v_n\}^\perp = \mathrm{Nul}\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix} = \mathrm{Nul}(A^T)$$

$$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$$

$$\xrightarrow{\text{Take } (\cdot)^\perp} \quad \mathrm{Col}(A) = (\mathrm{Col}(A)^\perp)^\perp = \mathrm{Nul}(A^T)^\perp$$

$$\xrightarrow[\text{by } A^T]{\text{replace } A} \quad \mathrm{Row}(A) = \mathrm{Col}(A^T) = \mathrm{Nul}(A)^\perp$$

and $$\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$$

**Orthogonality of the Four Subspaces:**

$$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T) \qquad \mathrm{Nul}(A^T)^\perp = \mathrm{Col}(A)$$

$$\mathrm{Nul}(A)^\perp = \mathrm{Row}(A) \qquad \mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$$

This says the two row picture subspaces $\mathrm{Row}(A)$, $\mathrm{Nul}(A)$ are orthogonal complements, & the two column picture subspaces $\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$ are orthogonal complements.

Eg: $V = \left\{ x \in \mathbb{R}^3 : \begin{matrix} x+2y = z \\ x+y+z=0 \end{matrix} \right\}$. Find a basis for $V^\perp$.

Step 0: $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow V^\perp = \mathrm{Row}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$V^\perp = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$: no elimination needed!

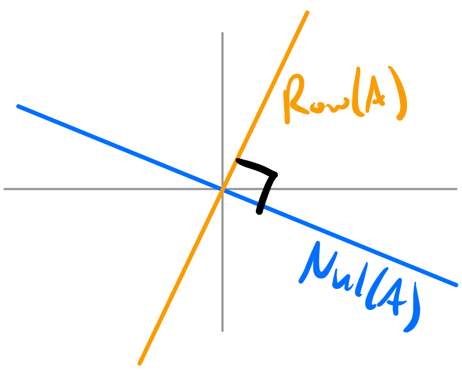
Eg: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$\rightsquigarrow \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Nul}(A^T) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Row Picture




Column Picture