# Solving Systems of Equations using Elimination

Here's a system of 3 equations in 3 variables:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases}$$

## How to solve it?

- **Substitution**: solve 1st equation for $x_1$, substitute into 2nd & 3rd, continue.
- **Elimination**: "combine" the equations to eliminate variables.

Elimination turns out to scale much better (to more equations & variables), so we'll focus on that.

"replace the 2nd equation with the 2nd minus 2x the 1st"

**Eg:**

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \quad \xrightarrow{R_2 -= 2R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$$\xrightarrow{R_3 -= 3R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -5x_2 - 10x_3 &= -20 \end{aligned}$$

Now we have eliminated $x_1$ from the 2nd & 3rd eq-s

These now form 2 equations in 2 variables: simpler!

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -5x_2 - 10x_3 &= -20 \end{aligned} \quad \xrightarrow{R_3 - = \frac{5}{7}R_2} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -\tfrac{50}{7}x_3 &= -\tfrac{150}{7} \end{aligned}$$

isolated

We eliminated $x_2$ from the last equation: now it's one equation in one variable. Easy!

We can now solve via back-substitution:

$$-\frac{50}{7}x_3 = -\frac{150}{7} \Longrightarrow \boxed{x_3 = 3}.$$

Substitute into 2nd equation:

$$-7x_2 - 4x_3 = 2 \rightsquigarrow -7x_2 - 4\cdot 3 = 2$$

isolated

Now solve for $x_2$:

$$-7x_2 - 12 = 2 \Longrightarrow -7x_2 = 14 \Longrightarrow \boxed{x_2 = -2}$$

Substitute both into 1st equation:

$$x_1 + 2x_2 + 3x_3 = 6 \rightsquigarrow x_1 + 2\cdot(-2) + 3\cdot 3 = 6$$

isolated

Now solve for $x_1$:

$$x_1 - 4 + 9 = 6 \Longrightarrow \boxed{x_1 = 1}$$

Check:

$$\begin{aligned} 1 + 2(-2) + 3(3) &= 6 \\ 2\cdot 1 - 3(-2) + 2(3) &= 14 \\ 3\cdot 1 + (-2) - 3 &= -2 \end{aligned} \quad \checkmark$$

NB: In this case there was one solution — since we could isolate each variable, all values were determined.

Does this always work?

Eg: 

$$4x_2 + 3x_3 = 2$$
$$x_1 + x_2 - x_3 = 3$$
$$2x_1 - 3x_2 - 6x_3 = -3$$

swap

$x_1$ is already eliminated from $R_1$. Fix: swap the 1st 2 eqns.

$$\xrightarrow{R_1 \leftrightarrow R_2} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ 2x_1 - 3x_2 - 6x_3 &= -3 \end{aligned}$$

Now eliminate as before:

$$\xrightarrow{R_3 -= 2R_1} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ -5x_2 - 4x_3 &= -9 \end{aligned}$$

$$\xrightarrow{R_3 += \frac{5}{4}R_2} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ -\tfrac{1}{4}x_3 &= -\tfrac{13}{2} \end{aligned}$$

Solve using back-substitution:

isolated $\quad -\frac{1}{4}x_3 = -\frac{13}{2} \implies x_3 = 26$

Substitute into 2nd equation:

isolated $\quad 4x_2 + 3(26) = 2 \implies x_2 = -19$

Substitute both into 1st equation:

isolated $\quad x_1 - 19 - 26 = 3 \implies x_1 = 48$

NB again there is one solution: each variable was isolated in one equation.

Check: $4x_2 + 3x_3 = 2$ $\quad 4(-19)+3(26)=2$

$x_1 + x_2 - x_3 = 3 \leadsto 48 - 19 - 26 = 3$

$2x_1 - 3x_2 - 6x_3 = -3 \quad 2(48) - 3(-19) - 6(26) = -3$ ✓

Eg: $x_1 + 2x_2 + 3x_3 = 1$
$4x_1 + 5x_2 + 6x_3 = 0$
$7x_1 + 8x_2 + 9x_3 = -1$

$\xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1}$

$x_1 + 2x_2 + 3x_3 = 1$
$-3x_2 - 6x_3 = -4$
$-6x_2 - 12x_3 = -8$

can't isolate $x_3$!

$\xrightarrow{R_3 -= 2R_2}$

$x_1 + 2x_2 + 3x_3 = 1$
$-3x_2 - 6x_3 = -4$
$0 = 0$

Are we done? Yes: choose any value for $x_3$, then back-substitute to find $x_1, x_2$:

$-3x_2 = -4 + 6x_3 \Rightarrow x_2 = \frac{4}{3} - 2x_3$

$x_1 = 1 - 2x_2 - 3x_3 = 1 - \frac{8}{3} + 4x_3 - 3x_3$

$x_1 = -5/3 + x_3$

Eg: $x_3 = 1 \leadsto x_1 = -2/3, \; x_2 = -2/3$

Check: $-2/3 - 4/3 + 3 = 1$
$-8/3 - 10/3 + 6 = 0$
$-14/3 - 16/3 + 9 = -1$ ✓

In this case there are infinitely many solutions. We'll deal with this in Week 3.

Eg: $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= 0 \end{aligned} \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ -6x_2 - 12x_3 &= -7 \end{aligned}$

tweak previous example

$\xrightarrow{R_3 -= 2R_2} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 &= 1 \end{aligned}$

If our original equations were true, then $0=1$.
Thus our system has no solutions.
(last 2 eqns are parallel planes)

Row Operations are the allowed manipulations we can perform on our equations.

(1) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_2 -= 2R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$

row replacement
replace $R_2$ by $R_2 - 2R_1$

(2) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_1 \leftrightarrow R_2} \begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 14 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$

row swap
(change order)

(3) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_1 \times = 2} \begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 12 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$

scalar multiplication
(by nonzero scalar)

Obviously if $(x_1, x_2, x_3)$ is a solution before doing a row operation, then it is true after. Eg. row replacement:

$$\begin{aligned} x_1+2x_2+3x_3 &\to 6 = 6 \\ 2x_1-3x_2+2x_3 &\to 14 = 14 \end{aligned} \xrightarrow{R_2 -= 2R_1} \begin{aligned} x_1+2x_2+3x_3 &\to 6 = 6 \\ -7x_2-4x_3 &\to 2 = 2. \end{aligned}$$

Fact: All these operations are reversible: if you have a solution $(x_1, x_2, x_3)$ after doing a row operation, then it's also a solution before.

(system of equations) $\xrightarrow[\text{operations}]{\text{row}}$ ((simpler) system of equations)

SAME SOLUTIONS

This was the whole point: we wanted to solve our (original) system of equations!

Questions: How do you undo (reverse):

- $R_1 += R_2$? $R_1 -= R_2$
- $R_1 \times= 2$? $R_1 \div= 2$
- $R_1 \leftrightarrow R_2$? $R_1 \leftrightarrow R_2$

The variables $x_1, x_2, \ldots$ are just placeholders; only their coefficients matter. Let's extract them into a matrix.

## Three Ways to Write System of Linear Equations

(1) As a system of equations:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \end{aligned}$$

(2) As a matrix equation $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 14 \end{bmatrix}}_{b}$$

If you expand out the product you get

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

which is what we had before.

The coefficient matrix $A$ comes from the coefficients of the variables:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix} \longleftrightarrow \begin{matrix} 1x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{matrix}$$

The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ contains the unknowns or variables.

NB: $A$ is an $m \times n$ matrix where

$m$ = # equations $\qquad b \in \mathbb{R}^m \leftarrow$ size $m$

$n$ = # variables $\qquad x \in \mathbb{R}^n \leftarrow$ size $n$

(3) As an augmented matrix.

This is a notational convenience: just squash $A$ & $b$ together and separate with a line.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \longleftrightarrow \begin{array}{l} 1x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \end{array}$$

$$\parallel$$

$$[A \mid b]$$

Augmented matrices are good for row operations, which only affect the coefficients (not the variables):

$$\begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \end{array} \xrightarrow{R_2 -= 2R_1} \begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ \quad -7x_2 - 4x_3 = 2 \end{array}$$

$$\parallel\!|$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \xrightarrow{R_2 -= 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \end{array}\right]$$

Eg: Let's solve the system from before using augmented matrices:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \xrightarrow{R_2 -= 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 -= 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$$

$$\xrightarrow{R_3 -= \frac{5}{7}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

$$\rightsquigarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ -7x_2 - 4x_3 = 2 \\ -\frac{50}{7}x_3 = -\frac{150}{7} \end{cases}$$

Now use back-substitution like before.

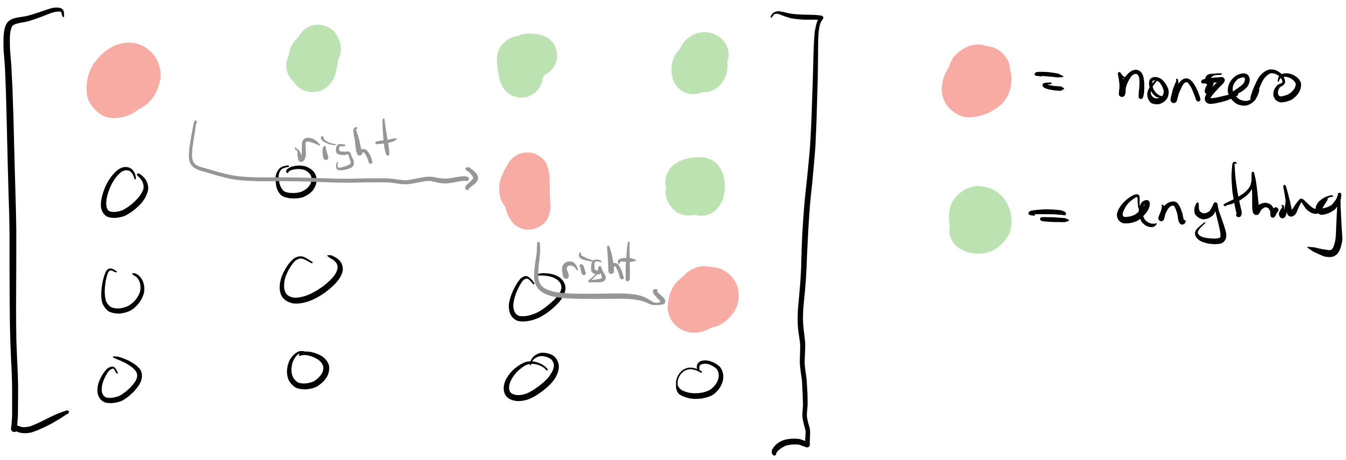
# What does it mean to be "done"?

(in terms of augmented matrices)

**Def:** A matrix is in **row echelon form (REF)** if

(1) The first nonzero entry of each row is to the <u>right</u> of the row above it

(2) All zero rows are at the bottom




REF: $\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$

Not REF: $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

**Important:** When checking if an augmented matrix is in REF, ignore the augmentation line.

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right]$ ← delete REF? $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix}$ ✓

**Think:** REF means there's nothing left to eliminate! Each variable is eliminated in later equations, or can't be isolated.

**Upshot:** The elimination procedure terminates when your (augmented) matrix is in REF.

Solving a linear system $\equiv$ Putting an augmented matrix into REF using row operations

**Def:** The pivot positions (pivots) of a matrix are the positions of the $1^{st}$ nonzero entries of each row after you put it into REF.

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

= pivots

Remarkably, this is well-defined!

**Def:** The rank of a matrix is the number of pivots it has (in REF).

**Eg:**

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \underset{(p.9)}{\xrightarrow{\text{REF}}} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

rank = 3

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{bmatrix} \underset{(p.4)}{\xrightarrow{\text{REF}}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2

## Number of Solutions (in terms of pivots)

The most basic question you can ask about a system of equations is: how many solutions does it have? This is entirely determined by the pivot positions/pivot columns (columns with a pivot).

(1) The system

pivot columns

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

(p.2)

had one solution. It has a pivot in every column except the augmented column. This means every variable will be isolated when doing back-substitution.

(0) The system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(p.5)

had no solutions. It has a pivot in the augmented column, which leads to the equation $0=1$.

$(\infty)$ The system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(p. 4)

had infinitely many solutions. It has no pivot in the augmented column and no pivot in the column for the variable $x_3$. You can't isolate $x_3$, so you can choose any value.

NB: You have to put the system in REF to find its pivots, so you have to do work to know how many solutions there are.

Def: A system is consistent if it has at least 1 solution (so 1 or $\infty$). It is inconsistent otherwise.