Linear Independence

Last time: we discussed subspaces, which are linear spaces thru 0, and two ways of describing them:

(1) As a Span/Col space

(2) As a solution set/Nul space

Today we focus on (1). In particular, we ask: when are we using too many vectors to span a given subspace?

Eg: (HW)

$$\mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix},\begin{pmatrix}2\\-5\\1\end{pmatrix},\begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$$ is a plane.

Why a plane and not $\mathbb{R}^3$? The vectors are coplanar: one is in the span of the others.

$$\frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix}-3\begin{pmatrix}2\\-5\\1\end{pmatrix}=\begin{pmatrix}-1\\5\\12\end{pmatrix}$$ [demo]

Any two non-collinear vectors span a plane:

$$\mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix},\begin{pmatrix}2\\-5\\1\end{pmatrix},\begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}=\mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix},\begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$$

not scalar multiples

This reduces the number of parameters needed to describe this subspace:

$$x_1\begin{pmatrix}2\\-4\\6\end{pmatrix}+x_2\begin{pmatrix}2\\-5\\1\end{pmatrix}+x_3\begin{pmatrix}-1\\5\\12\end{pmatrix} \quad \text{vs.} \quad x_1\begin{pmatrix}2\\-4\\6\end{pmatrix}+x_2\begin{pmatrix}-1\\5\\12\end{pmatrix}$$

Moreover, the expression with 2 parameters is unique, but with 3 parameters it is redundant:

$$1\begin{pmatrix}2\\-4\\6\end{pmatrix}-1\begin{pmatrix}2\\-5\\1\end{pmatrix}+0\begin{pmatrix}-1\\5\\12\end{pmatrix}=\begin{pmatrix}0\\1\\5\end{pmatrix}=6\begin{pmatrix}2\\-4\\6\end{pmatrix}-7\begin{pmatrix}2\\-5\\1\end{pmatrix}-2\begin{pmatrix}-1\\5\\12\end{pmatrix}$$

but $\begin{pmatrix}0\\1\\5\end{pmatrix}=x_1\begin{pmatrix}2\\-4\\6\end{pmatrix}+x_2\begin{pmatrix}2\\-5\\1\end{pmatrix}$ only for

[demo] $x_1=1,\ x_2=-1$

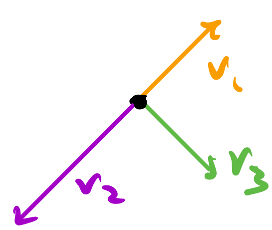
We want to formalize this notion that there are "too many" vectors spanning this subspace by saying one is in the span of the others.

In the above example, each vector is in the span of the other 2, but this need not be the case.

Eg: $v_1=\begin{pmatrix}1\\1\end{pmatrix}$ $v_2=\begin{pmatrix}-2\\-2\end{pmatrix}$ $v_3=\begin{pmatrix}1\\-1\end{pmatrix}$

Here $v_2=-2v_1+0v_3$

but $v_3\notin \text{Span}\{v_1,v_2\}$



We want a condition that means some vector is in the span of the others. Answer: rewrite as a homogeneous vector equation.

Def: A list of vectors $\{v_1,\ldots,v_n\}$ is linearly dependent (LD) if the vector equation

$$x_1v_1+\cdots+x_nv_n=0$$

has a nontrivial solution. Such a solution is called a linear relation among $\{v_1,\ldots,v_n\}$

Eg:

move here

$$\begin{pmatrix}-1\\5\\12\end{pmatrix} = \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

$$\rightsquigarrow\ 0 = -\begin{pmatrix}-1\\5\\12\end{pmatrix} + \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

is a linear relation

$$\Longrightarrow \left\{\begin{pmatrix}-1\\5\\12\end{pmatrix}, \begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\} \text{ is LD}$$

move here

$$v_2 = -2v_1 + 0v_3 \ \rightsquigarrow\ 0 = -2v_1 - v_2 + 0v_3$$

is a linear relation

$$\Longrightarrow \{v_1, v_2, v_3\} \text{ is LD}$$

Recall: $Ax=0$ has a nontrivial solution

$\iff$ A has a free variable

(otherwise the only solution is $x=0$)

$\{v_1,\dots,v_n\}$ is LD

$\iff$ $x_1v_1+\cdots+x_nv_n=0$ has a nontrivial solution

$\iff$ the matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ has a free variable

NB: If $x_1v_1+\cdots+x_nv_n=0$ and $x_i\neq 0$ then

$$v_i=-\frac{1}{x_i}\left(x_1v_1+\cdots+x_{i-1}v_{i-1}+x_{i+1}v_{i+1}+\cdots+x_nv_n\right)$$

so v_i is in the span of the others.

LD means some vector is in the span of the others: $x_1v_1+\cdots+x_nv_n=0$ and $x_i\neq 0$ implies $v_i\in \operatorname{Span}\{v_1,\dots,v_{i-1},v_{i+1},\dots,v_n\}$

Summary: Let $v_1, \dots, v_n$ be vectors.

The following are equivalent:

(1) $\{v_1, \dots, v_n\}$ is linearly dependent

(2) The matrix

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$ has a free variable

(3) Some v_i is in the span of the others

Def: A list of vectors $\{v_1, \dots, v_n\}$ is linearly independent (LI) if it is not linearly dependent: ie, if the vector equation

$$x_1 v_1 + \cdots + x_n v_n = 0$$

has only the trivial solution.

A logically equivalent statement is:

$$x_1 v_1 + \cdots + x_n v_n = 0 \quad \text{implies} \quad x_1 = \cdots = x_n = 0.$$

The logical negation of the Summary above is:

Summary: Let $v_1, \ldots, v_n$ be vectors.
The following are equivalent:

(1) $\{v_1, \ldots, v_n\}$ is linearly independent

(2) The matrix

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$ does not have a free variable

(3) No v_i is in the span of the others

Roughly, vectors $v_1, \ldots, v_n$ are LI if their span is as large as it can be. Every time you add a vector, the span gets bigger!

Eg: Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ LI or LD?

In other words, does the vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

have a nontrivial solution? free $\Rightarrow$ LD

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{PF}} \begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Take $x_3 = 1 \rightsquigarrow$ linear relation

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

So they're LD [demo]

Eg: Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ -8 \\ 9 \end{pmatrix} \right\}$ LI or LD?

In other words, does the vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ -8 \\ 9 \end{pmatrix} = 0$$

have a nontrivial solution?

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix} \underset{\text{REF}}{\rightsquigarrow} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -22 \\ 0 & 0 & 32 \end{bmatrix}$$

No free variables $\Rightarrow$ only the trivial solution

$\Rightarrow$ these vectors are LI [demo]

Fact: If $\{v_1, \dots, v_n\}$ is LI and $b \in \mathrm{Span}\{v_1, \dots, v_n\}$ then there are unique weights $x_1, \dots, x_n$ such that

$$b = x_1 v_1 + \cdots + x_n v_n$$

In other words, this is not a redundant parameterization of $\mathrm{Span}\{v_1, \dots, v_n\}$

Proof: Let A be the matrix with cols $v_1, \dots, v_n$

so $Ax = b \quad \equiv \quad x_1 v_1 + \cdots + x_n v_n = b$

$Ax = b$ is consistent because $b \in \mathrm{Col}(A)$

$\Rightarrow Ax = b$ has one soln because A have no free variables. //

Linguistic note: LI, LD are adjectives that apply to a set of vectors.

Bad: "A is LI" "v_1 is LD on v_2 and v_3"

Good: "A has LI columns" "$\{v_1, v_2, v_3\}$ is LD"

Eg:
- $\{v\}$ is LI $\iff$ $v \neq 0$
- Any set containing the 0 vector is LD: if $v_1 = 0$ then
$$0 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + v_n$$
is a linear relation.
- Suppose $\{v, w\}$ is LD. So there exist $(a,b) \neq (0,0)$ such that $av + bw = 0$.
$$\left.\begin{array}{l} a \neq 0 \rightsquigarrow v = -\frac{b}{a} w \\ b \neq 0 \rightsquigarrow w = -\frac{a}{b} v \end{array}\right\} \text{ } v, w \text{ are collinear.}$$

$$\boxed{\{v, w\} \text{ is LD} \iff v, w \text{ are collinear.}}$$

- Similarly, $\{u, v, w\}$ is LD $\iff$ u, v, w are coplanar, and so on.
- If $r > n$ then r vectors in $\mathbb{R}^n$ are LD: the matrix $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{bmatrix}$ is wide, so it has a free variable.

eg. 3 vectors in $\mathbb{R}^2$ are automatically LD. [demo]

Basis and Dimension

A basis of a subspace is a minimal set of vectors needed to span (parameterize/describe) that subspace.

Def: A set of vectors $\{v_1, \ldots, v_n\}$ is a basis for a subspace V if:

(1) $V = \mathrm{Span}\{v_1, \ldots, v_n\}$

(2) $\{v_1, \ldots, v_n\}$ is linearly independent

The dimension of V is the number of vectors in any basis. (Fact: all bases have the same size!)

Notation: $\dim(V)$

Spans means you get a parameterization of V:

$$b \in V \implies b = x_1 v_1 + \cdots + x_n v_n$$

LI means this parameterization is unique.

Rephrase: A spanning set for V is a basis if it is linearly independent.

Eg: $V = \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$

A basis is $\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \right\}$. (or $\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$)

(1) Spans: because $\begin{pmatrix}-1\\5\\12\end{pmatrix} \in \text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$

(2) LI: because not collinear.

So $\dim(V) = 2$ (a plane) ✓

Eg: $\{0\} = \text{Span}\{\} \Rightarrow \dim\{0\} = 0$ ✓

Eg: A line L is spanned by one vector

$\Rightarrow \dim(L) = 1$.

In general:

- A point has dimension 0
- A line has dimension 1
- A plane has dimension 2

etc.

Eg: What is a basis for $\mathbb{R}^n$?

The unit coordinate vectors $e_1, \dots, e_n$.

$n=3$: $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$x_1e_1 + x_2e_2 + x_3e_3 = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

(1) Spans: every vector has this form.

(2) LI: if this $= 0$ then $x_1 = x_2 = x_3 = 0$ ✓

So $\dim(\mathbb{R}^n) = n$ ✓

NB: $\mathbb{R}^n$ has many bases.

eg. $\mathbb{R}^2$ is spanned by any pair of noncollinear vectors: $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$; $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$; $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right\}$, ...

In fact, any nonzero subspace has infinitely many bases! Parameterizations are not unique!

→ A basis is a way to describe a subspace using the fewest vectors possible.

Warning: Be careful to distinguish between these:

Subspace	Basis	Matrix
$V = \mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$	$\begin{pmatrix}2 & 2\\-4 & -5\\6 & 1\end{pmatrix}$

This is a subspace. It is a plane. It has ∞ vectors in it.

This is a matrix A. Its columns form a basis for $V = \mathrm{Col}\,A$.

This is a basis for V. It has 2 vectors in it. It is a finite list of data that describes V.

Bases for Col(A) & Null(A)

Remember, if someone hands you a subspace, you want to describe it as a column space or a null space so you can do computations, like find a basis.

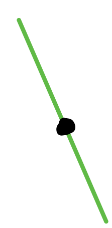
Thm: The pivot columns of A form a basis of Col(A).

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{basis: } \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

↑pivot column

NB: Take the pivot columns of the original matrix, Not the RREF. Doing row ops changes the column space!

$$\mathrm{Col}\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$



$$\mathrm{Col}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

Proof: Let R be the RREF of A.

$$A = \begin{bmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the pivot columns are v_1, v_2, v_4.

Note: $Ax=0 \iff Rx=0$ (same solution set)

(1) **Spans:** $\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow 0 = -3\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 4 \\ 6 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow R\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow A\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow v_3 = 3v_1 + 2v_2$$

$$\boxed{\text{A and R have the same col relations!}}$$

Similarly, $\begin{pmatrix} 4 \\ 6 \\ -1 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow v_5 = 4v_1 + 6v_2 - v_4$$

Any vector in $\mathrm{Col}(A)$ has the form

$$v = x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 + x_5 v_5$$
$$= x_1 v_1 + x_2 v_2 + x_3(3v_1 + 2v_2) + x_4 v_4 + x_5(4v_1 + 6v_2 - v_4)$$
$$= (x_1 + 3x_3 + 4x_5)v_1 + (x_2 + 2x_3 + 6x_5)v_2 + (x_4 - x_5)v_4$$

which is in $\operatorname{Span}\{v_1, v_2, v_4\}$.

(2) LI: If $x_1 v_1 + x_2 v_2 + x_4 v_4 = 0$ then

$$A\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0 \Rightarrow R\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 = x_4 = 0 \checkmark$$

Consequence: The number of vectors in a basis for Col(A) is equal to the number of pivots of A.

$$\boxed{\operatorname{rank}(A) = \dim \operatorname{Col}(A)}$$

Eg: Find a basis for Span $\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$

Step 0: Rewrite as Col $\begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix}$

Now find pivot columns:

$$\begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix}2 & 2 & -1\\0 & -1 & 3\\0 & 0 & 0\end{pmatrix}$$

Basis: $\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$

2 pivots $\rightsquigarrow$ Span is a plane.

Thm: The vectors attached to the free variables in the parametric vector form of the solution set of $Ax=0$ form a basis for Nul(A)

$$\begin{bmatrix}1 & 2 & 2 & 1\\2 & 4 & 1 & -1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1 & 2 & 0 & -1\\0 & 0 & 1 & 1\end{bmatrix}$$

$$\xrightarrow{\text{PVF}} x = x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

basis: $\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\right\}$

Proof:

(1) Spans: Every solution $= x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$ ✓

(2) LI: Think about it in parametric form:

$$\begin{aligned}
0 = x_1 &= -2x_2 + x_4 \\
0 = x_2 &= x_2 \\
0 = x_3 &= -x_4 \\
0 = x_4 &= x_4
\end{aligned}
\quad \Rightarrow \quad x_2 = x_4 = 0 \checkmark$$

trivial equations

Consequence:

$$\dim \mathrm{Nul}(A) = \#\text{free vars} = \#\text{cols} - \text{rank}$$

NB: This is consistent with our provisional definition of the dimension of a solution set as the number of free variables.