The Four Subspaces

Recall: To any matrix A, we can associate:

- $\mathrm{Col}(A)$; basis = pivot columns of A; dim = rank
- $\mathrm{Nul}(A)$; basis = vectors in the PVF of $Ax=0$; dim = # free vars = # cols − rank

There are two more subspaces: just replace A by A^T, then take Col & Nul.

Why? Orthogonality ⇝ least □s (bear with me...)

Def: The row space of A is $\mathrm{Row}(A) = \mathrm{Col}(A^T)$.

This is the subspace spanned by the rows of A, regarded as (row) vectors in $\mathbb{R}^n$.

This is a subspace of $\mathbb{R}^n$ $\quad n = \#$ columns
($n = \#$ entries in each row)

⇝ row picture

Is a span ⇝ parametric description

Eg:

$$\mathrm{Row}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\} = \mathrm{Col}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Fact: Row operations do not change the row space.

Why? If the rows are v_1, v_2, v_3 then $\mathrm{Row}(A) = \mathrm{Span}\{v_1, v_2, v_3\}$. Row ops:

- $R_1 \longleftrightarrow R_3$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_3, v_2, v_1\}$
- $R_2 \times= 3$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, 3v_2, v_3\}$
- $R_2 += 2R_1$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2 + 2v_1, v_3\}$

  because $v_2 + 2v_1 \in \mathrm{Span}\{v_1, v_2, v_3\}$

  and $v_2 = (v_2 + 2v_1) - 2v_1 \in \mathrm{Span}\{v_1, v_2 + 2v_1, v_3\}$ ✓

This is a col space (of A^T), so you know how to compute a basis (pivot columns of A^T). But you can also find a basis by doing elimination on A:

Thm: The nonzero rows of any REF of A form a basis for $\mathrm{Row}(A)$.

Eg: $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \end{pmatrix} \right\}$

or: $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(another) **Basis:** $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Proof:

(1) Spans: row ops don't change Row(A), and you can always delete the zero vector without changing the span

(2) LI: $0 = x_1 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \\ 2x_1 - 3x_2 \\ x_1 - 3x_2 \end{pmatrix}$

Solve by forward-substitution:

● = pivot, so this entry in the sum is just
$(1) \cdot x_1 = 0 \Rightarrow x_1 = 0$

● = pivot, so this entry in the sum is just
$(-3) x_2 = 0 \Rightarrow x_2 = 0$ ✓

Consequence: $\dim \mathrm{Row}(A) = \#$ pivot rows
$= \#$ pivots $=$ rank.

(a nonzero row of an REF matrix has a pivot)

Def: The **left null space** of A is $\mathrm{Nul}(A^T)$.

This is the solution set of $A^Tx=0$.

Notation: just $\mathrm{Nul}(A^T)$ (no new notation)

This is a subspace of $\mathbb{R}^m$ $\quad m=$ # rows

($m=$ # columns of A^T)

$\rightsquigarrow$ column picture

is a null space $\rightsquigarrow$ implicit description

NB: $A^Tx=0 \iff 0=(A^Tx)^T=x^TA$

so $\mathrm{Nul}(A^T)=\{$ row vectors $x^T\in\mathbb{R}^m :\ x^TA=0\}$

$\mathrm{Nul}(A^T)$ is a null space, so you know how to compute a basis (PVF of $A^Tx=0$). You can also find a basis by doing elimination on A:

Thm/Procedure: To compute a basis of $\mathrm{Nul}(A^T)$:

(1) Form the augmented matrix $(A \mid I_m)$

(2) Eliminate to REF

(3) The rows on the right side of the line next to zero rows on the left form a basis of $\mathrm{Nul}(A^T)$.

Eg: $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix}$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 -= R_1]{R_2 -= 2R_1} \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 -= R_2} \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

zero row   basis

Basis for $\mathrm{Nul}(A^T)$: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Check: $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$

so at least $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in \mathrm{Nul}(A^T)$ ✓

Consequence:

$$\dim \mathrm{Nul}(A^T) = m - r = \#\text{rows} - \text{rank}$$

Proof of the Thm: Suppose $A \xrightarrow{REF} U$. Then

$U = E \cdot A \qquad E =$ product of elementary matrices

$$\Rightarrow E \cdot (A \mid I_m) = (EA \mid EI_m)$$
$$= (U \mid E)$$

So the result of performing elimination on $(A|I_m)$ is $(U|E)$.

If U is in REF and the last $m-r$ rows are zero then we claim:

$$\mathrm{Nul}(U^T) = \mathrm{Span}\{e_{r+1}, e_{r+2}, \dots, e_m\}:$$

We know $U^T e_i$ = the i^{th} row of U.

We know from before that the nonzero rows of U are LI. So if $(x_1, \dots, x_m) \in \mathrm{Nul}(U^T)$ then

$$0 = U^T \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1 U^T e_1 + \cdots + x_r U^T e_r + x_{r+1} U^T e_{r+1} + \cdots + x_m U^T e_m$$

These are 0 because the last $m-r$ rows of U are 0.

$$= x_1 U^T e_1 + \cdots + x_r U^T e_r = 0$$

This implies $x_1 = \cdots = x_r = 0$ because the first r rows of U are LI

So $0 = U^T(x_1, \dots, x_m) \iff x_1 = \cdots = x_r = 0$

$\iff (x_1, \dots, x_m) \in \mathrm{Span}\{e_{r+1}, e_{r+2}, \dots, e_m\}$

This proves the claim.

Now, $U = EA \implies U^T = A^T E^T$, so

$$A^T E^T x = 0 \implies U^T x = 0$$

$$\iff x = a_{r+1} e_{r+1} + a_{r+2} e_{r+2} + \cdots + a_m e_m$$

But $E^T e_i$ is the i^{th} row of E, so

$$E^T x = a_{r+1} E^T e_{r+1} + a_{r+2} E^T e_{r+2} + \cdots + a_m E^T e_m$$

= a LC of the last $m-r$ rows of E

so $A^T E^T x = 0$

$\iff E^T x \in \text{Span}\{\text{last } m-r \text{ rows of } E\}$.

(I've left out some details at the end) //

NB: The left null space is changed by row operations:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \qquad \text{Nul}(A^T) = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$\Big\downarrow$$

$$U = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{Nul}(U^T) = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Summary: Four Subspaces

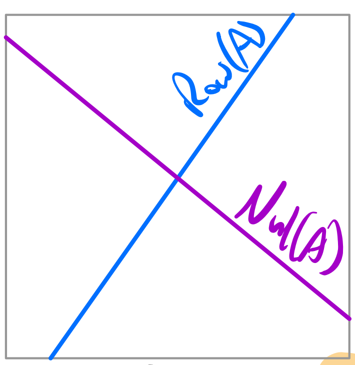
A : an $m \times n$ matrix of rank r

Subspace	of	row/col	dim	basis	implicit/parametric
$\mathrm{Col}(A)$	$\mathbb{R}^m$	col	r	pivot cols of A	parametric
				# pivot cols	
$\mathrm{Nul}(A)$	$\mathbb{R}^n$	row	$n-r$	vectors in PVF	implicit
				# free vars	
$\mathrm{Row}(A)$	$\mathbb{R}^n$	row	r	nonzero rows of REF	parametric
				# pivot rows	
$\mathrm{Nul}(A^T)$	$\mathbb{R}^m$	col	$m-r$	last $m-r$ rows of E	implicit
				# zero rows in REF	

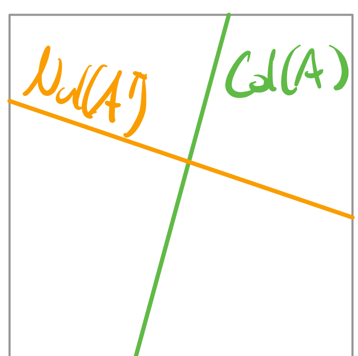
The row picture subspaces ($\mathrm{Nul}(A)$, $\mathrm{Row}(A)$) are unchanged by row operations

The col picture subspaces ($\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$) are changed by row operations.

dim $\mathrm{Row}(A)$ + dim $\mathrm{Nul}(A)$ $= n$



row picture $\mathbb{R}^n$



col picture $\mathbb{R}^m$

dim $\mathrm{Col}(A)$ + dim $\mathrm{Nul}(A^T)$ $= m$

♪♪ The row space lives in the... row picture!
The null space lives in the... row picture!
The other two live in the... column picture!
That's how you keep them straight.

Consequences:

$$\boxed{\begin{array}{c}\text{Row Rank = Column Rank} \\ \dim \mathrm{Row}(A) = \mathrm{rank} = \dim \mathrm{Col}(A)\end{array}}$$

So A & A^T have the same # pivots — in completely different positions! (HW#5)

$$\boxed{\begin{array}{c}\text{Rank-Nullity} \\ \dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = n = \#\text{cols} \\ \dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A^T) = m = \#\text{rows}\end{array}}$$

[demos]

NB: You can compute bases for all four subspaces by doing elimination once.

$$A \rightsquigarrow [A \mid I_m] \rightsquigarrow [\mathrm{RREF}(A) \mid E]$$

- Get the pivots of $A \rightsquigarrow \mathrm{Col}(A)$
- Get $\mathrm{RREF}(A) \rightsquigarrow$ PVF of $Ax=0 \rightsquigarrow \mathrm{Nul}(A)$
- Get nonzero rows of $\mathrm{RREF}(A) \rightsquigarrow \mathrm{Row}(A)$
- Get rows of $E \rightsquigarrow \mathrm{Nul}(A^T)$

Full-Rank Matrices

A "random" matrix will have largest rank possible.
This is an important special case.

Def: An $m \times n$ matrix A of rank r has:

- full column rank if $r = n$ eg. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
  (every column has a pivot)
- full row rank if $r = m$ eg. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
  (every row has a pivot)

NB: Each row & column has at most one pivot
so $r \leq \min\{m, n\}$

Hence full row/column rank means full rank
ie. largest possible rank.

NB: A has full column rank $\Rightarrow n = r \leq m$
$\Rightarrow A$ is tall (at least as many rows as cols)
A has full row rank $\Rightarrow m = r \leq n$
$\Rightarrow A$ is wide (at least as many cols as rows)

We've seen several properties of matrices that translate into "there's a pivot in every column".

Thm: The Following Are Equivalent (TFAE):

(for a given matrix A, all are true or all are false)

(1) A has full column rank

(1') A has a pivot in every column

(1'') A has no free columns.

(2) $\mathrm{Nul}(A) = \{0\}$

(2') $Ax = 0$ has only the trivial solution.

(2'') $Ax = b$ has 0 or 1 soln for every $b \in \mathbb{R}^m$

(3) The columns of A are LI

(4) $\dim \mathrm{Col}(A) = n$

(5) $\dim \mathrm{Row}(A) = n$

(5') $\mathrm{Row}(A) = \mathbb{R}^n$ ← simple description of Row(A)!

NB: (5) ⟺ (5') because:

$\boxed{\text{The only } n\text{-dimensional subspace of } \mathbb{R}^n \text{ is all of } \mathbb{R}^n}$

Eg: There is no plane in $\mathbb{R}^2$ that doesn't fill up all of $\mathbb{R}^2$.

We've seen several properties of matrices that translate into "there's a pivot in every row".

Thm: TFAE:

(1) A has full row rank

(1') A has a pivot in every row

(1'') A REF of A has no zero rows

(2) $\dim \mathrm{Col}(A) = m$

(2') $\mathrm{Col}(A) = \mathbb{R}^m$ ← simple description of Col(A)!

(2'') $Ax=b$ is consistent for every $b \in \mathbb{R}^m$
↳ (has 1 or ∞ solutions)

(3) The columns of A span $\mathbb{R}^m$

(4) $\dim \mathrm{Row}(A) = m$

(5) $\mathrm{Nul}(A^T) = \{0\}$

Again, (2)⟺(2') because the only m-dimensional subspace of $\mathbb{R}^m$ is all of $\mathbb{R}^m$.

If A has full column rank and full row rank then

$n = r = m$

$\Rightarrow A$ is square and has n pivots: invertible.

Thm: For an $n \times n$ matrix A, TFAE:

(1) A is invertible

(2) A has full column rank

(3) A has full row rank

(4) $RREF(A) = I_n$

(5) There is a matrix B with $AB = I_n$

(6) There is a matrix B with $BA = I_n$

(7) $Ax = b$ has exactly one solution for every b

namely, $x = A^{-1}b$

(8) A^T is invertible

(row rank = col rank)

Consequence: Let $\{v_1, \dots, v_n\}$ be vectors in $\mathbb{R}^n$

$\rightsquigarrow A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ is an $n \times n$ matrix.

(1) $\text{Span}\{v_1, \dots, v_n\} = \mathbb{R}^n \iff \text{Col}(A) = \mathbb{R}^n$

$\iff A$ has FRP

$\iff A$ is invertible

(1) $\{v_1, \dots, v_n\}$ is LI

$\iff Ax = 0$ has only the trivial soln

$\iff A$ has FCR

$\iff A$ is invertible

Of course, (1) + (2) means $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$, so

$$\left(\begin{matrix} \text{basis for} \\ \mathbb{R}^n \end{matrix} \right) \equiv \left(\begin{matrix} \text{columns of an} \\ \text{invertible } n \times n \text{ matrix} \end{matrix} \right)$$

More on this next time (Basis Theorem).