Vector and Matrix Algebra

These basic definitions don't need to be done in a live lecture — that's why I'm recording this one.

Since we'll be doing algebra with vectors & matrices as well as numbers, we give "numbers" a new name to distinguish them.

"definition"

Def: A scalar is a real number.

"is an element of"

Notation: $c \in \mathbb{R}$ ← the set of all real numbers

"example"

Eg: $2, -\pi, e^{\sqrt{3}}, 0 \in \mathbb{R}$

Def: A vector is a finite (ordered) list of numbers

The size of a vector is the length of the list.
The numbers in the list are the coordinates.

"is an element of"

Notation: $v \in \mathbb{R}^n$ ← "the set of all lists of n numbers" $\quad n =$ the size of v

Eg: $v = \begin{pmatrix} 2 \\ -\frac{\pi}{4} \\ e^{\sqrt{3}} \end{pmatrix} \in \mathbb{R}^3$ (size 3) $\qquad w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4$ (size 4)

coordinates

"note"

NB: We will usually write vectors in a column like $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (a "column vector") but this is just notation; $v = (1,2,3)$ means the same thing.

NB: Some people decorate vectors with

boldface: $\mathbf{v}$

arrow: $\vec{v}$

but I won't do that since it's annoying and it's usually clear from context which letters represent vectors.

Important Example:

The unit coordinate vectors in $\mathbb{R}^n$ are vectors with one coordinate $= 1$ and the rest $= 0$.

Notation: $\boxed{e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n}$

This notation is fixed for the whole semester.

The size of e_i must be infered from context.

In $\mathbb{R}^3$ the unit coordinate vectors are:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eg: The zero vector is the vector

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

(again the size must be infered from context)

Def: Two vectors are equal if they have the same size and the same coordinates.

Eg: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ since the sizes are different.

Vector Algebra

You can multiply a vector by a scalar:

Scalar Multiplication:

$$c \in \mathbb{R},\quad v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \rightsquigarrow c \cdot v = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \in \mathbb{R}^n$$

(scalar) $\times$ (vector) = (vector)

Eg: $2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 \\ 2\cdot 2 \\ 2\cdot 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ $\quad 0\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0\cdot 1 \\ 0\cdot 2 \\ 0\cdot 3 \end{pmatrix} = \mathbf{0}$

$$-\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (-1)\cdot\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

You can add & subtract vectors componentwise:

Vector Addition & Subtraction:

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\rightsquigarrow \quad u \pm v = \begin{pmatrix} x_1 \pm y_1 \\ \vdots \\ x_n \pm y_n \end{pmatrix} \in \mathbb{R}^n$$

(vector) ± (vector) = (vector)

NB: You can only add/subtract vectors of the same size.

Eg: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} \pi \\ e \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1+\pi \\ 2+e \\ 3+\sqrt{2} \end{pmatrix}$ $\qquad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \times$

You can "multiply" two vectors, but you get a scalar:

Dot Product / Inner Product:

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\rightsquigarrow \quad u \cdot v = x_1 y_1 + \cdots + x_n y_n \in \mathbb{R}$$

(vector) · (vector) = (scalar)

NB: You can only dot vectors of the same size.

Eg: $\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\4\end{pmatrix} = 1\cdot 2 + 2\cdot(-1) + 3\cdot 4 = 12$

Eg: $\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\2\end{pmatrix} = 1(-2)+2(-1)+2(2) = 0$

Eg: $\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\cdot1+2\cdot2+3\cdot3 = 14$

NB: If $v=(x_1,\dots,x_n)$ then $v\cdot v = x_1^2+\dots+x_n^2$.

This is a nonnegative number; it is $=0$

$\iff v=0$.

"if and only if"

$\rightarrow$ Do not write $\cancel{v^2 = v\cdot v}$!

(You can't take any other "powers" of a vector: $v^3 = v\cdot v\cdot v$ doesn't make sense because $v\in\mathbb{R}^n$ and $v\cdot v\in\mathbb{R}$.)

Rules for Vector Algebra: $c\in\mathbb{R}$ $u,v,w\in\mathbb{R}^n$

(1) $c(u\pm v) = cu \pm cv$ (distributivity over scalar $\times$)

(2) $u\cdot(cv) = c(u\cdot v) = (cu)\cdot v$ (associativity of $\cdot$)

(3) $u\cdot v = v\cdot u$ (commutativity of $\cdot$)

(4) $u\cdot(v\pm w) = u\cdot v \pm u\cdot w$ (distributivity over $\cdot$)

Eg: if $u, v, x, y \in \mathbb{R}^n$ then

$$(u+v)\cdot(x+y) \overset{(4)}{=} (u+v)\cdot x + (u+v)\cdot y$$
$$\overset{(3,4)}{=} u\cdot x + v\cdot x + u\cdot y + v\cdot y$$

Upshot: FOIL works fine.

You can add and scalar multiply at the same time:

Def: A **linear combination** of vectors $v_1, \ldots, v_n \in \mathbb{R}^m$ with **weights** $x_1, \ldots, x_n \in \mathbb{R}$ is the vector

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \in \mathbb{R}^m$$

Eg: If $v = (x_1, \ldots, x_m) \in \mathbb{R}^m$ then

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_m \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_m e_m$$

The coordinates of v became the weights of this linear combination of unit coordinate vectors.

Matrix Algebra

Def: A matrix is a box holding a 2D grid of numbers.
The size of a matrix is (#rows) × (#cols).

> We usually (but not always) write
> $$m = \#\text{rows} \qquad n = \#\text{cols}$$
> so A is an $m \times n$ matrix.

The (i,j)-entry of A is the number in the i^{th} row and j^{th} column.

Eg: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$ is a 3×4 matrix.

The (i,j)-entry is a_{ij}.

(sometimes I'll use square brackets – it means the same)

Eg: $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ is a 3×2 matrix.

The $(3,2)$-entry is 6.

Def: The diagonal entries of a matrix are the (i,j)-entries for $i = j$:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

● = diagonal entries

Def: A matrix is diagonal if all non-diagonal entries are zero:

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \end{pmatrix}$$

● = any number

Def: A matrix is square if (# rows) = (# columns):

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

3×3 ⤳ square

Eg: The $n \times n$ identity matrix is

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{pmatrix}$$

This is a square, diagonal matrix.
Its columns are $e_1, \ldots, e_n$. (So are its rows.)

Eg: The $m \times n$ zero matrix is

$$0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

This is a diagonal matrix because the non-diagonal entries are zero.

(so are the diagonal entries)

Scalar Multiplication & Matrix Addition/Subtraction

are again done componentwise:

$$c\cdot\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \\ ca_{31} & ca_{32} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} \\ a_{31}\pm b_{31} & a_{32}\pm b_{32} \end{pmatrix}$$

They satisfy distributivity: $c(A\pm B) = cA \pm cB$

NB: You can only add/subtract matrices of the same size.

NB: A vector of size n is just an $n\times 1$ matrix

Def: A row vector is a matrix with one row.

$$u = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \qquad 1\times 3 \rightsquigarrow \text{row vector}$$

You can multiply a matrix & a vector:

Matrix × Vector = Vector:

There are 2 ways to compute this:

(1) By Columns: If A has columns $v_1, \dots, v_n \in \mathbb{R}^m$ then

$$Ax = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + \cdots + x_n v_n \in \mathbb{R}^m$$

for $x \in \mathbb{R}^n$.

> The coordinates of x are the weights of the columns of A in a linear combination.

$\boxed{A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ is the linear combination of the columns of } A \text{ with weights } x_1, \dots, x_n}$

(2) By Rows: If A has rows $w_1, \dots, w_m \in \mathbb{R}^n$ then

$$Ax = \begin{pmatrix} — & w_1 & — \\ & \vdots & \\ — & w_m & — \end{pmatrix} x = \begin{pmatrix} w_1 \cdot x \\ \vdots \\ w_m \cdot x \end{pmatrix}$$

dot products

$\boxed{\text{The } i^{\text{th}} \text{ coordinate of } Ax \text{ is } (\text{row } i) \cdot x}$

NB: You get the same answer either way! (#2 is probably easier by hand)

Eg: $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \overset{\#1}{=} 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \overset{\#2}{=} \begin{pmatrix} 1\cdot 2 + 4\cdot(-1) \\ 2\cdot 2 + 5\cdot(-1) \\ 3\cdot 2 + 6\cdot(-1) \end{pmatrix} = \begin{pmatrix} 2-4 \\ 4-5 \\ 6-6 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

NB: Ax only makes sense when the size of x equals the number of columns of A:

$$\boxed{A \text{ is } m \times n \rightsquigarrow \quad x \in \mathbb{R}^n \qquad Ax \in \mathbb{R}^m}$$

$$\underset{(m\times n)}{A} \cdot \underset{(n\times 1)}{x} \rightsquigarrow \underset{(m\times 1)}{Ax}$$

Eg: $A \cdot e_i \overset{\#1}{=} 0\cdot(\text{col } 1) + \cdots + 1\cdot(\text{col } i) + \cdots + 0\cdot(\text{col } n)$

$= \text{col } i$

$$\boxed{Ae_i = \text{the } i^{\text{th}} \text{ column of } A}$$

Eg: $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{e_2} = 0\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 0\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

identity matrix: p.8

Eg: $I_n x = \begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\#1}{=} x_1 e_1 + \cdots + x_n e_n \overset{p.6}{=} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x$

$$\boxed{I_n x = x \quad \text{for all } x}$$

Matrix × Matrix = Matrix: Column Form

Let A be an $m \times n$ matrix.

Let B be an $n \times p$ matrix with columns $u_1, \ldots, u_p \in \mathbb{R}^n$

The product AB is the $m \times p$ matrix with columns $Au_1, \ldots, Au_p \in \mathbb{R}^m$:

$$AB = A \begin{pmatrix} | & & | \\ u_1 & \cdots & u_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Au_1 & \cdots & Au_p \\ | & & | \end{pmatrix}$$

NB: This only makes sense if (#cols of A) = (#rows of B)

$$\begin{array}{ccccc} A & & B & & AB \\ (m \times n) & \times & (n \times p) & \rightsquigarrow & m \times p \end{array}$$

NB: You can compute the Au_i using #1 or #2.

Using #2: if A has rows $w_1, \dots, w_m$ then

$$Au_i = (w_1 \cdot u_i, \dots, w_m \cdot u_i) \quad \text{so}$$

$$\begin{pmatrix} - & w_1 & - \\ - & w_2 & - \\ & \vdots & \\ - & w_m & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_p \\ | & | & & | \end{pmatrix} = \begin{pmatrix} w_1 \cdot u_1 & w_1 \cdot u_2 & \cdots & w_1 \cdot u_p \\ w_2 \cdot u_1 & w_2 \cdot u_2 & \cdots & w_2 \cdot u_p \\ \vdots & \vdots & \ddots & \vdots \\ w_m \cdot u_1 & w_m \cdot u_2 & \cdots & w_m \cdot u_p \end{pmatrix}$$

$$\boxed{\text{So } (i,j) \text{ entry of } AB = (\text{row } i \text{ of } A) \cdot (\text{row } j \text{ of } B)}$$

Eg: Compute $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$.

$$2 \times 3 \qquad 3 \times 2 \quad \rightsquigarrow \quad 2 \times 2$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix} \overset{\#1}{=} \left(1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 3 \\ -4 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix} \overset{\#2}{=} \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 4 & 1\cdot 3 + 2\cdot 1 + 3\cdot(-1) \\ -1\cdot 1 + 2\cdot 2 + (-4)\cdot 4 & -1\cdot 3 + 2\cdot 1 + (-4)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix}$$

identity matrix: p.8

Eg: $$A \boxed{I_n} = A\begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ae_1 & \cdots & Ae_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ (1^{st} \text{ col of } A) & \cdots & (n^{th} \text{ col of } A) \\ | & & | \end{pmatrix} = A$$

Eg: $$I_m A = I_m \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ I_m v_1 & \cdots & I_m v_n \\ | & & | \end{pmatrix} \overset{p.12}{=} \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = A$$

$$\boxed{I_m A = A = A I_n}$$

Def: A column vector times a row vector is called an **outer product**:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \end{pmatrix}$$

$$2 \times 1 \qquad 1 \times 3 \quad \rightsquigarrow \quad 2 \times 3$$

Eg: $$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \overset{\#1}{=} \left(3\begin{pmatrix} 1 \\ 2 \end{pmatrix} \; 4\begin{pmatrix} 1 \\ 2 \end{pmatrix} \; 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \end{pmatrix}$$

Matrix × Matrix = Matrix: Outer Product Form

Let A be an $m \times n$ matrix.

with columns $v_1, \dots, v_n \in \mathbb{R}^m$

Let B be an $n \times p$ matrix

with rows $w_1, \dots, w_n \in \mathbb{R}^p$

The product AB is the $m \times p$ matrix

$$AB = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} — & w_1 & — \\ & \vdots & \\ — & w_n & — \end{pmatrix}$$

$$= v_1 w_1^T + v_2 w_2^T + \cdots + v_n w_n^T$$

(w_i^T means write w_i as a row vector)

Eg:

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} \begin{pmatrix} 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 12 & -3 \\ -16 & 4 \end{pmatrix} = \begin{pmatrix} 17 & 2 \\ -13 & 3 \end{pmatrix} \checkmark$$

There is one more algebraic operation on matrices:

Def: Let A be an $m \times n$ matrix. Its **transpose** is the matrix A^T whose rows are the columns of A (& vice-versa)

The (i,j)-entry of A is the (j,i)-entry of A^T:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$(1,2)$-entry $\qquad\qquad$ $(2,1)$-entry

You can think of transposing as "reflecting over the diagonal"

Eg: $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Eg: If $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ $\quad w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ then

$$v^T w = (x_1 \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (x_1 y_1 + \cdots + x_n y_n) = v \cdot w$$

(a 1×1 matrix is a scalar)

$$\boxed{v^T w = v \cdot w \quad \text{for } v, w \in \mathbb{R}^n}$$

Compare:

$v^T w =$ inner product (scalar)

$v w^T =$ outer product (matrix)

Def: A matrix is symmetric if it equals its transpose: $A = A^T$

NB: symmetric matrices are square!

Eg: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ is symmetric

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$ is not: $A^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

Eg: Let A be an $m \times n$ matrix with columns $v_1, \ldots, v_n \in \mathbb{R}^m$. Then A^T is $n \times m$ $\rightsquigarrow$ $A^T A$ is $n \times n$.

$$A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$$

$$A^T A = \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \cdots & v_n \cdot v_n \end{pmatrix}$$

Since $v_i \cdot v_j = v_j \cdot v_i$, this is symmetric.

$\boxed{\text{$A^TA$ is the matrix of column dot products. It is symmetric.}}$

The matrix A^TA will play a huge role in the 2^{nd} half of the course.

Rules for Matrix Algebra:

Let A, B, C be matrices. Assume all sizes are compatible in what follows.

(1) $A(BC) = (AB)C$ (associativity)

$\rightarrow$ Thus it makes sense to write ABC (evaluate AB or BC first $\rightarrow$ same thing)

$\rightarrow$ If $C = v$ is a vector then $A(Bv) = (AB)v$

(2) $A(B \pm C) = AB \pm AC$
$(A \pm B)C = AC \pm BC$ (distributivity)

(3) $I_m A = A = AI_n$ (identity)

(4) $A(cB) = c(AB) = (cA)B$ for $c \in \mathbb{R}$. (scalars)

(5) $(A^T)^T = A$ (double transpose)

(6) $(A \pm B)^T = A^T \pm B^T$ (transpose & sums)

(7) $(AB)^T = \cancel{A^TB^T}\ B^TA^T$ (transpose & products)

NB: A^TB^T may not even make sense:

$A: m\times n \quad B: n\times p$

$\rightsquigarrow AB: (m\times n)\cdot(n\times p) = (m\times p)$

$A^TB^T: (n\times m)\cdot(p\times n) = ??$

$B^TA^T: (p\times n)\cdot(n\times m) = (p\times m)$ ✓

Eg: $(A^TA)^T \overset{(6)}{=} A^T(A^T)^T \overset{(4)}{=} A^TA$

$\Rightarrow A^TA$ is symmetric (again)

Eg: If A is square then AA makes sense, as do $A\cdot A\cdot A$, $A\cdot A\cdot A\cdot A$, etc.

Def: If A is square then its nth power $(n>0)$ is

$$A^n = A\cdots A \quad (n \text{ times})$$

This only makes sense because of associativity.

Question: What about A^{-1}? (week 3)

Caveats:

Commutativity fails in general:

$$AB \neq BA$$

Eg: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

$$\nparallel$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

Cancellation fails in general:

$$A \neq 0,\ AB = AC \not\Rightarrow B = C$$

Eg: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$