MATH 218D-1
FINAL EXAMINATION

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic. You may bring a 8.5×11**-inch note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

Problem 1. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 5 & -2 \\ 4 & 1 & 3 & -2 \end{pmatrix}.$$

Compute the singular value decomposition of A in outer product form.

$$A = 4\sqrt{3}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 2 & -1 \end{pmatrix} + 2\sqrt{3}\,\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} 2 & 1 & -1 & 0 \end{pmatrix}$$

Problem 2.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 5 & 1 & 8 \\ 6 & 8 & -8 & -4 \\ -3 & 1 & 5 & 10 \end{pmatrix}.$$

The matrix $A^T A$ has orthogonal diagonalization $A^T A = QDQ^T$ where

$$Q = \begin{pmatrix} -2/\sqrt{30} & 1/\sqrt{15} & 8/3\sqrt{10} & 2/3\sqrt{5} \\ -1/\sqrt{30} & 3/\sqrt{15} & -1/3\sqrt{10} & -4/3\sqrt{5} \\ 3/\sqrt{30} & -1/\sqrt{15} & 5/3\sqrt{10} & -4/3\sqrt{5} \\ 4/\sqrt{30} & 2/\sqrt{15} & 0 & 1/\sqrt{5} \end{pmatrix} \qquad D = \begin{pmatrix} 270 & 0 & 0 & 0 \\ 0 & 135 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) The rank of A is $r = \boxed{2}$.

b) The SVD of A in matrix form is $A = U\Sigma V^T$ where:

$$U = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 3\sqrt{30} & 0 & 0 & 0 \\ 0 & 3\sqrt{15} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad V = Q$$

c) The *maximum* value of $\|Ax\|$ subject to $\|x\| = 1$ is $\boxed{3\sqrt{30}}$.

d) The *minimum* value of $\|Ax\|$ subject to $\|x\| = 1$ is $\boxed{0}$.

e) The *minimum* value of $\|A^T x\|$ subject to $\|x\| = 1$ is $\boxed{0}$.

Problem 3. [20 points]

The following centered data matrix contains six samples of four measurements each:

$$A = \begin{pmatrix} -6 & 3 & -2 & 1 & 2 & 2 \\ -1 & -3 & 0 & 4 & 1 & -1 \\ 1 & 0 & -1 & -4 & 2 & 2 \\ -4 & 7 & 1 & 2 & -3 & -3 \end{pmatrix}.$$

The covariance matrix is

$$S = \frac{1}{5}\begin{pmatrix} 58 & 1 & 0 & 33 \\ 1 & 28 & -17 & -9 \\ 0 & -17 & 26 & -25 \\ 33 & -9 & -25 & 88 \end{pmatrix}.$$

a) The measurement variances are

$$s_1^2 = \boxed{58/5} \qquad s_2^2 = \boxed{28/5} \qquad s_3^2 = \boxed{26/5} \qquad s_4^2 = \boxed{88/5}.$$

b) The total variance is $s^2 = \boxed{40}$.

c) The variance along the subspace $\mathbf{R}^4$ is $s(\mathbf{R}^4)^2 = \boxed{40}$.

d) The variance in the direction $u = \frac{1}{\sqrt{2}}(0, 1, 1, 0)$ is $s(u)^2 = \boxed{2}$.

e) The variance along the plane

$$V = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{is} \quad s(V)^2 = \boxed{58/5}.$$

f) The variance along $V^\perp$ is $s(V^\perp)^2 = \boxed{142/5}$.

The covariance matrix has an orthogonal diagonalization $S = UDU^T$ where

$$U \approx \begin{pmatrix} -0.49 & -0.61 & 0.59 & -0.21 \\ 0.036 & 0.45 & 0.69 & 0.57 \\ 0.23 & -0.63 & -0.2 & 0.72 \\ -0.84 & 0.2 & -0.37 & 0.34 \end{pmatrix} \qquad D \approx \begin{pmatrix} 23 & 0 & 0 & 0 \\ 0 & 9.3 & 0 & 0 \\ 0 & 0 & 7.7 & 0 \\ 0 & 0 & 0 & .13 \end{pmatrix}.$$

g) For $k = 1,2,3$, find an orthonormal basis for the k-space of best fit V_k, and find the error^2 in the sense of variance of the orthogonal distance from V_k:

V_1: basis: $\left\{ \begin{pmatrix} -.49 \\ .036 \\ .23 \\ -.84 \end{pmatrix} \right\}$ $\qquad s(V_1^\perp)^2 = \boxed{17.13}$

V_2: basis: $\left\{ \begin{pmatrix} -.49 \\ .036 \\ .23 \\ -.84 \end{pmatrix}, \begin{pmatrix} -.61 \\ .45 \\ -.63 \\ .2 \end{pmatrix} \right\}$ $\qquad s(V_2^\perp)^2 = \boxed{7.83}$

V_3: basis: $\left\{ \begin{pmatrix} -.49 \\ .036 \\ .23 \\ -.84 \end{pmatrix}, \begin{pmatrix} -.61 \\ .45 \\ -.63 \\ .2 \end{pmatrix}, \begin{pmatrix} .59 \\ .69 \\ -.2 \\ -.37 \end{pmatrix} \right\}$ $\qquad s(V_3^\perp)^2 = \boxed{.13}$

h) Which subspace V_1, V_2, V_3 is the most useful approximation for your data set, and why?

The most useful approximation is V_3: it is a very good fit for the data (the error^2 is small), and it still reduces the dimension of the data set from 4 to 3.

Problem 4.

[15 points]

A certain matrix A has a factorization $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 3 & -5 & 2 \\ -2 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) The trace and determinant of A are

$$\mathrm{Tr}(A) = \boxed{3} \qquad \det(A) = \boxed{-3}.$$

b) Compute C^{-1}. (Its entries are all whole numbers.)

$$C^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 5 \end{pmatrix}$$

c) $A^k \begin{pmatrix} -7 \\ 8 \\ -3 \end{pmatrix} = 3^k \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}$

d) If $v \in \mathbf{R}^3$, then $\|A^k v\| \to \infty$ unless $v \in \mathrm{Span}\left\{ \begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$.

Problem 5. [15 points]

Consider the quadratic form

$$q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 - 2x_1x_2 + 8x_1x_3 + 8x_2x_3$$

corresponding to the symmetric matrix

$$S = \begin{pmatrix} 2 & -1 & 4 \\ -1 & 2 & 4 \\ 4 & 4 & 5 \end{pmatrix}.$$

a) The maximum value of $q(x)$ subject to $\|x\| = 1$ is 9. Find a unit vector u_1 such that $q(u_1) = 9$.

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

b) The minimum value of $q(x)$ subject to $\|x\| = 1$ is -3. Find a unit vector u_3 such that $q(u_3) = -3$.

$$u_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

c) Compute the maximum value of $q(x)$ subject to $\|x\| = 1$ and $x \cdot u_1 = 0$, and find a unit vector u_2 achieving this value. [Hint: use cross products!]

$$u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad q(u_2) = \boxed{3}$$

d) Find an orthogonal matrix Q such that q becomes diagonal after changing coordinates $x = Qy$.

$$Q = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{3} & 0 \end{pmatrix}$$

e) Find a matrix A such that $S = A^T A$, or explain why no such matrix exists.

No such matrix exists because S is indefinite.

Problem 6. [15 points]

This problem contains short, unrelated computations.

a) Find the singular value decomposition of $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ in outer product form.

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

b) $\lim_{k\to\infty} \begin{pmatrix} .8 & .1 & .2 \\ .1 & .8 & .2 \\ .1 & .1 & .6 \end{pmatrix}^k \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

c) If $V = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} : x, z \in \mathbf{R} \right\}$ (this is the xz-plane in $\mathbf{R}^3$) then $P_V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) Is this symmetric matrix positive-definite? ◯ Yes ● No

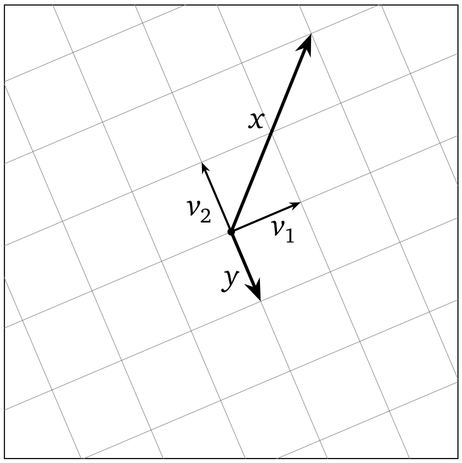
$$S = \begin{pmatrix} 3 & 6 & -3 \\ 6 & 11 & -7 \\ -3 & -7 & 3 \end{pmatrix}$$

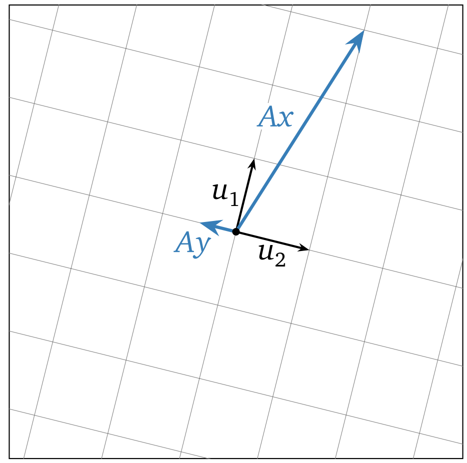
Problem 7. [15 points]

a) A certain 2×2 matrix A has the singular value decomposition

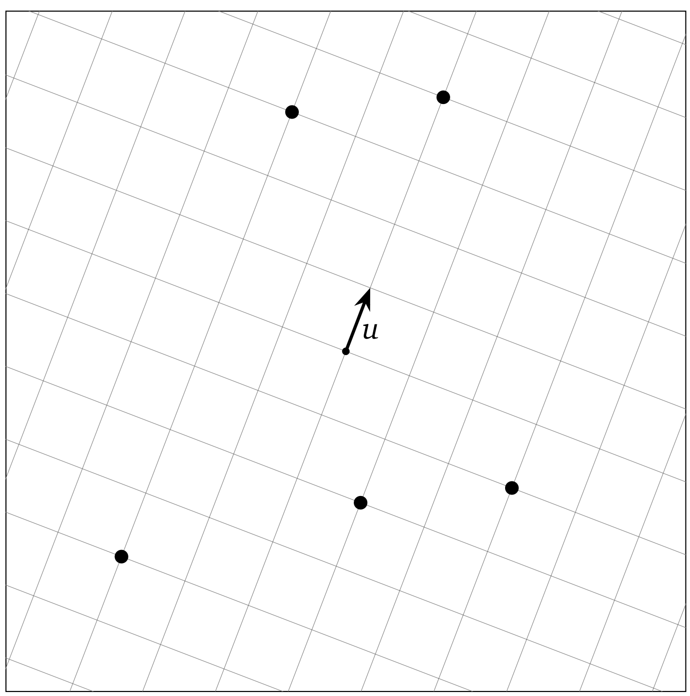
$$A = \frac{3}{2} u_1 v_1^T + \frac{1}{2} u_2 v_2^T,$$

where u_1, u_2, v_1, v_2 are drawn in the diagrams below. Given x and y in the diagram on the left, draw Ax and Ay on the diagram on the right.





b) The columns of a certain 2×5 centered data matrix A are drawn as dots in the diagram below. Compute the variance in the u-direction, where u is the vector in the diagram. (Grid marks are one unit apart.)



$s(u)^2 =$ 23/2

Problem 8.

[20 points]

Short-answer questions: no justification is necessary unless otherwise indicated.

a) Suppose that A is a 3×3 matrix such that $Ax = \begin{pmatrix}1\\2\\-1\end{pmatrix}$ has exactly one solution. Which of the following can you conclude about A? Fill in the bubbles of all correct answers.

- ● A has full row rank
- ● $Ax = \begin{pmatrix}2\\-3\\0\end{pmatrix}$ has exactly one solution
- ○ $\det(A) = 0$
- ● A has full column rank
- ● A is invertible
- ○ A is diagonalizable
- ● $Ax = 0$ has a solution

b) Which of the following matrices are diagonalizable using real and/or complex numbers?

- ● $\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$
- ○ $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$
- ● $\begin{pmatrix}1 & 3 & 2\\ 0 & 2 & 1\\ 0 & 0 & 3\end{pmatrix}$
- ● $\begin{pmatrix}1 & 3 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{pmatrix}$
- ● $\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$
- ● P_V for $V = \mathrm{Nul}\begin{pmatrix}1 & -3 & 2\\ -2 & 6 & -4\end{pmatrix}$

c) Suppose that A is a 2×3 matrix with singular value decomposition $A = 3u_1v_1^T + 2u_2v_2^T$.

$\det(A^TA) =$ 0

$\det(AA^T) =$ 36

$\mathrm{Tr}(A^TA) =$ 13

$\mathrm{Tr}(AA^T) =$ 13

d) Which of the following subspaces are equal to $V = \{(x, y, z) : x + y + z = 0\}$? Fill in the bubbles of all correct answers.

- ● $\mathrm{Nul}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$
- ● $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}^{\perp}$
- ● $\mathrm{Col}\begin{pmatrix}3 & 1 & -1\\ -2 & -2 & -2\\ -1 & 1 & 3\end{pmatrix}$
- ○ $\mathrm{Row}\begin{pmatrix}3 & 2 & 1\\ 1 & 2 & 1\end{pmatrix}$
- ○ $\mathrm{Nul}\begin{pmatrix}1 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}^T$
- ● $\mathrm{Nul}\begin{pmatrix}2 & 2 & 2\\ -1 & -1 & -1\end{pmatrix}$

e) If A is a positive stochastic matrix, explain why the steady state w is in $\mathrm{Col}(A)$.

By definition we have $Aw = w$, so the equation $Ax = w$ is consistent, which means $w \in \mathrm{Col}(A)$.

f) If A is a 30×40 matrix of rank 25, then $\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A) =$ 40.

Problem 9. [20 points]

In each part either provide an example, or briefly *explain* why no such example exists. *All matrices have real entries.*

a) A 2×2 matrix of rank 1 with infinitely many singular value decompositions.

This is impossible. In order for a matrix to have infinitely many singular value decompositions, it must have a repeated singular value, but a matrix of rank 1 has only one singular value.

b) A 2×2 matrix such that $\mathrm{Nul}(A) = \mathrm{Row}(A)$.

This is impossible because the null space and the row space are orthogonal complements.

c) A 2×2 matrix with singular values $\sigma_1 = 2$ and $\sigma_2 = 1$, and left singular vectors $u_1 = \frac{1}{\sqrt{5}}\binom{1}{2}$ and $u_2 = \frac{1}{\sqrt{5}}\binom{-2}{1}$.

We can choose any right singular vectors that we like. For instance,

$$A = 2\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + 1\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}2 & -2\\4 & 1\end{pmatrix}.$$

d) A basis for $\mathbf{R}^3$ containing the vectors $(1,1,1)$ and $(1,1,-1)$.

There are many answers. One is

$$\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}.$$

e) A 2×2 matrix A such that A^TA is not diagonalizable.

This is impossible by the spectral theorem since A^TA is symmetric.

f) A 2×2 symmetric matrix that does not have a Cholesky decomposition.

You just have to write down a symmetric matrix that is not positive definite. For instance,

$$S = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$