

MATH 218D-1
FINAL EXAMINATION

| | | | |
|-------------|--|-------------------|--|
| Name | | Duke Email | |
|-------------|--|-------------------|--|

usually first.last@duke.edu

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic. You may bring a 8.5×11 -inch **note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

[This page intentionally left blank]

Problem 1.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 5 & -2 \\ 4 & 1 & 3 & -2 \end{pmatrix}.$$

Compute the singular value decomposition of A in outer product form.

$A =$

[Scratch work for Problem 1]

Problem 2.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 5 & 1 & 8 \\ 6 & 8 & -8 & -4 \\ -3 & 1 & 5 & 10 \end{pmatrix}.$$

The matrix $A^T A$ has orthogonal diagonalization $A^T A = Q D Q^T$ where

$$Q = \begin{pmatrix} -2/\sqrt{30} & 1/\sqrt{15} & 8/3\sqrt{10} & 2/3\sqrt{5} \\ -1/\sqrt{30} & 3/\sqrt{15} & -1/3\sqrt{10} & -4/3\sqrt{5} \\ 3/\sqrt{30} & -1/\sqrt{15} & 5/3\sqrt{10} & -4/3\sqrt{5} \\ 4/\sqrt{30} & 2/\sqrt{15} & 0 & 1/\sqrt{5} \end{pmatrix} \quad D = \begin{pmatrix} 270 & 0 & 0 & 0 \\ 0 & 135 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) The rank of A is $r =$.

b) The SVD of A in matrix form is $A = U \Sigma V^T$ where:

$$U = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \quad \Sigma = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \quad V = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

c) The *maximum* value of $\|Ax\|$ subject to $\|x\| = 1$ is .

d) The *minimum* value of $\|Ax\|$ subject to $\|x\| = 1$ is .

e) The *minimum* value of $\|A^T x\|$ subject to $\|x\| = 1$ is .

[Scratch work for Problem 2]

Problem 3.

[20 points]

The following centered data matrix contains six samples of four measurements each:

$$A = \begin{pmatrix} -6 & 3 & -2 & 1 & 2 & 2 \\ -1 & -3 & 0 & 4 & 1 & -1 \\ 1 & 0 & -1 & -4 & 2 & 2 \\ -4 & 7 & 1 & 2 & -3 & -3 \end{pmatrix}.$$

The covariance matrix is

$$S = \frac{1}{5} \begin{pmatrix} 58 & 1 & 0 & 33 \\ 1 & 28 & -17 & -9 \\ 0 & -17 & 26 & -25 \\ 33 & -9 & -25 & 88 \end{pmatrix}.$$

a) The measurement variances are

$$s_1^2 = \boxed{} \quad s_2^2 = \boxed{} \quad s_3^2 = \boxed{} \quad s_4^2 = \boxed{}.$$

b) The total variance is $s^2 = \boxed{}$.

c) The variance along the subspace \mathbf{R}^4 is $s(\mathbf{R}^4)^2 = \boxed{}$.

d) The variance in the direction $u = \frac{1}{\sqrt{2}}(0, 1, 1, 0)$ is $s(u)^2 = \boxed{}$.

e) The variance along the plane

$$V = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{is} \quad s(V)^2 = \boxed{}.$$

f) The variance along V^\perp is $s(V^\perp)^2 = \boxed{}$.

[Scratch work for Problem 3]

(Problem 3, continued)

The covariance matrix has an orthogonal diagonalization $S = UDU^T$ where

$$U \approx \begin{pmatrix} -0.49 & -0.61 & 0.59 & -0.21 \\ 0.036 & 0.45 & 0.69 & 0.57 \\ 0.23 & -0.63 & -0.2 & 0.72 \\ -0.84 & 0.2 & -0.37 & 0.34 \end{pmatrix} \quad D \approx \begin{pmatrix} 23 & 0 & 0 & 0 \\ 0 & 9.3 & 0 & 0 \\ 0 & 0 & 7.7 & 0 \\ 0 & 0 & 0 & .13 \end{pmatrix}.$$

- g) For $k = 1, 2, 3$, find an orthonormal basis for the k -space of best fit V_k , and find the error² in the sense of variance of the orthogonal distance from V_k :

$$V_1: \text{ basis: } \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right\} \quad s(V_1^\perp)^2 = \boxed{}$$

$$V_2: \text{ basis: } \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right\} \quad s(V_2^\perp)^2 = \boxed{}$$

$$V_3: \text{ basis: } \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right\} \quad s(V_3^\perp)^2 = \boxed{}$$

- h) Which subspace V_1, V_2, V_3 is the most useful approximation for your data set, and why?

[Scratch work for Problem 3]

Problem 4.

[15 points]

A certain matrix A has a factorization $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 3 & -5 & 2 \\ -2 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) The trace and determinant of A are

$$\text{Tr}(A) = \boxed{} \quad \det(A) = \boxed{}.$$

b) Compute C^{-1} . (Its entries are all whole numbers.)

$$C^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

c) $A^k \begin{pmatrix} -7 \\ 8 \\ -3 \end{pmatrix} =$

d) If $v \in \mathbb{R}^3$, then $\|A^k v\| \rightarrow \infty$ unless $v \in \text{Span} \left\{ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \right\}.$

[Scratch work for Problem 4]

Problem 5.

[15 points]

Consider the quadratic form

$$q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 - 2x_1x_2 + 8x_1x_3 + 8x_2x_3$$

corresponding to the symmetric matrix

$$S = \begin{pmatrix} 2 & -1 & 4 \\ -1 & 2 & 4 \\ 4 & 4 & 5 \end{pmatrix}.$$

- a) The maximum value of $q(x)$ subject to $\|x\| = 1$ is 9. Find a unit vector u_1 such that $q(u_1) = 9$.

$$u_1 = \begin{pmatrix} \\ \\ \end{pmatrix}$$

- b) The minimum value of $q(x)$ subject to $\|x\| = 1$ is -3 . Find a unit vector u_3 such that $q(u_3) = -3$.

$$u_3 = \begin{pmatrix} \\ \\ \end{pmatrix}$$

[Scratch work for Problem 5]

(Problem 5, continued)

- c) Compute the maximum value of $q(x)$ subject to $\|x\| = 1$ and $x \cdot u_1 = 0$, and find a unit vector u_2 achieving this value. [Hint: use cross products!]

$$u_2 = \begin{pmatrix} \\ \\ \end{pmatrix} \quad q(u_2) = \boxed{}$$

- d) Find an orthogonal matrix Q such that q becomes diagonal after changing coordinates $x = Qy$.

$$Q = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

- e) Find a matrix A such that $S = A^T A$, or explain why no such matrix exists.

[Scratch work for Problem 5]

Problem 6.

[15 points]

This problem contains short, unrelated computations.

- a) Find the singular value decomposition of $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ in outer product form.

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} =$$

b) $\lim_{k \rightarrow \infty} \begin{pmatrix} .8 & .1 & .2 \\ .1 & .8 & .2 \\ .1 & .1 & .6 \end{pmatrix}^k \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$

c) If $V = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} : x, z \in \mathbf{R} \right\}$ (this is the xz -plane in \mathbf{R}^3) then $P_V = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

- d) Is this symmetric matrix positive-definite? ☐ Yes ☐ No

$$S = \begin{pmatrix} 3 & 6 & -3 \\ 6 & 11 & -7 \\ -3 & -7 & 3 \end{pmatrix}$$

[Scratch work for Problem 6]

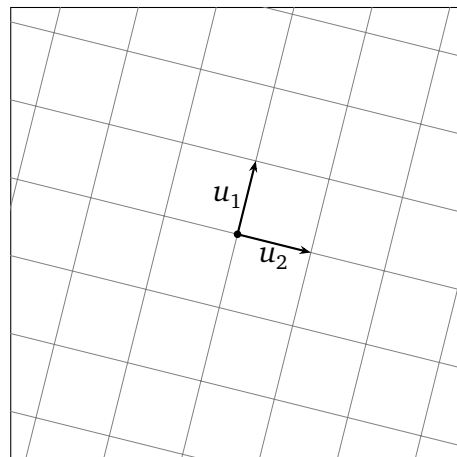
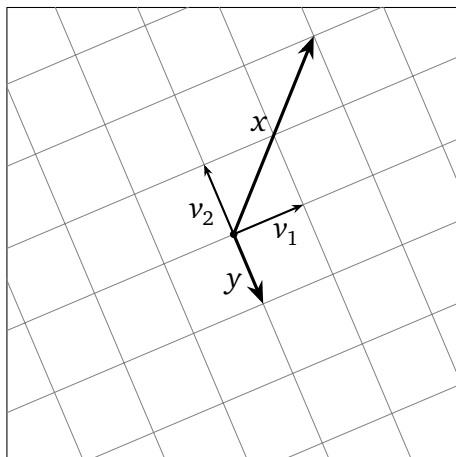
Problem 7.

[15 points]

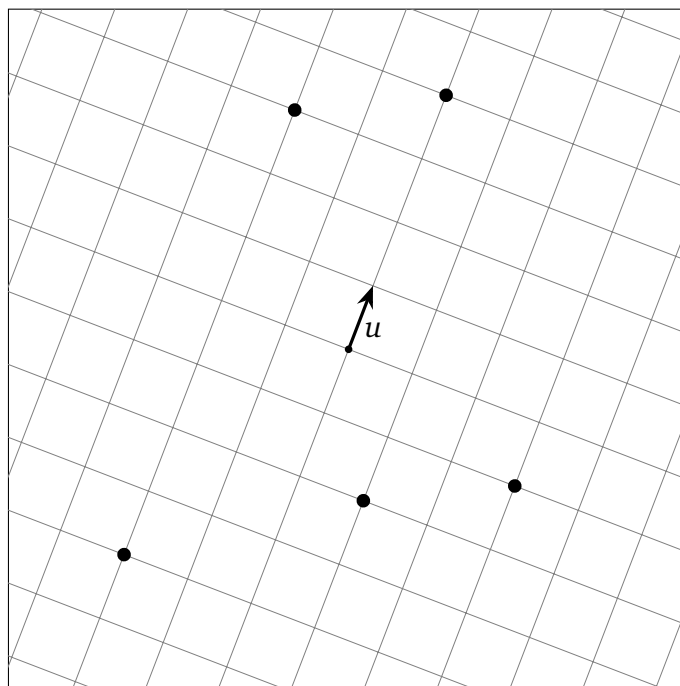
- a) A certain 2×2 matrix A has the singular value decomposition

$$A = \frac{3}{2}u_1v_1^T + \frac{1}{2}u_2v_2^T,$$

where u_1, u_2, v_1, v_2 are drawn in the diagrams below. Given x and y in the diagram on the left, draw Ax and Ay on the diagram on the right.



- b) The columns of a certain 2×5 centered data matrix A are drawn as dots in the diagram below. Compute the variance in the u -direction, where u is the vector in the diagram. (Grid marks are one unit apart.)



$$s(u)^2 = \boxed{}$$

[Scratch work for Problem 7]

Problem 8.

[20 points]

Short-answer questions: no justification is necessary unless otherwise indicated.

- a) Suppose that A is a 3×3 matrix such that $Ax = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ has exactly one solution. Which of the following can you conclude about A ? Fill in the bubbles of all correct answers.

- | | |
|--|--|
| <input type="radio"/> A has full row rank | <input type="radio"/> A has full column rank |
| <input type="radio"/> $Ax = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$ has exactly one solution | <input type="radio"/> A is invertible |
| <input type="radio"/> $\det(A) = 0$ | <input type="radio"/> A is diagonalizable |
| | <input type="radio"/> $Ax = 0$ has a solution |

- b) Which of the following matrices are diagonalizable using real and/or complex numbers?

- | | | | |
|--|--|---|---|
| <input type="radio"/> $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ | <input type="radio"/> $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | <input type="radio"/> $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ | <input type="radio"/> $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ |
| <input type="radio"/> $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | <input type="radio"/> P_V for $V = \text{Nul} \begin{pmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{pmatrix}$ | | |

- c) Suppose that A is a 2×3 matrix with singular value decomposition $A = 3u_1v_1^T + 2u_2v_2^T$.

$$\det(A^T A) = \boxed{}$$

$$\det(AA^T) = \boxed{}$$

$$\text{Tr}(A^T A) = \boxed{}$$

$$\text{Tr}(AA^T) = \boxed{}$$

[Scratch work for Problem 8]

(Problem 8, continued)

d) Which of the following subspaces are equal to $V = \{(x, y, z) : x + y + z = 0\}$? Fill in the bubbles of all correct answers.

☐ $\text{Nul} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ ☐ $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}^\perp$ ☐ $\text{Col} \begin{pmatrix} 3 & 1 & -1 \\ -2 & -2 & -2 \\ -1 & 1 & 3 \end{pmatrix}$

☐ $\text{Row} \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ ☐ $\text{Nul} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^T$ ☐ $\text{Nul} \begin{pmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}$

e) If A is a positive stochastic matrix, explain why the steady state w is in $\text{Col}(A)$.

f) If A is a 30×40 matrix of rank 25, then $\dim \text{Row}(A) + \dim \text{Nul}(A) = \boxed{}$.

[Scratch work for Problem 8]

[20 points]

a) A 2×2 matrix of rank 1 with infinitely many singular value decompositions.

b) A 2×2 matrix such that $\text{Nul}(A) = \text{Row}(A)$.

c) A 2×2 matrix with singular values $\sigma_1 = 2$ and $\sigma_2 = 1$, and left singular vectors $u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

d) A basis for \mathbf{R}^3 containing the vectors $(1, 1, 1)$ and $(1, 1, -1)$.

e) A 2×2 matrix A such that $A^T A$ is not diagonalizable.

f) A 2×2 symmetric matrix that does not have a Cholesky decomposition.

[Scratch work for Problem 9]