MATH 218D-1
PRACTICE MIDTERM EXAMINATION 1

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1. [10 points]

a) Compute the LU decomposition of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 & -2 \\ 1 & 1/2 & 2 & -1 \\ 2 & -2 & 4 & 0 \end{pmatrix}.$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & -1 & 2 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

b) Briefly describe under what circumstances you might want to compute the LU decomposition of a matrix in practice.

If you have to solve $Ax = b$ for many values of b, then it makes sense to only do elimination once.

Problem 2. [15 points]

Consider the matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -3 & -1 & -1 \\ 3 & -9 & 1 & 1 \\ 2 & -6 & 0 & -1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) $\operatorname{rank}(A) = \boxed{3}$.

b) Find a basis for each of the four fundamental subspaces of A.

Any nonzero subspace has infinitely many bases. You probably chose these. (Note that $\operatorname{Nul}(A^T) = \{0\}$ because A has rank 3.)

$$\operatorname{Nul}(A): \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \qquad \operatorname{Row}(A): \left\{ \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\operatorname{Col}(A): \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} \qquad \operatorname{Nul}(A^T): \{\}$$

c) Which of the following are true about A? Fill in the bubbles of all that apply.

- ● $\operatorname{Col}(A) = \mathbf{R}^3$
- ○ $\operatorname{Row}(A) = \mathbf{R}^4$
- ● $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent
- ○ $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{Nul}(A)$
- ● $Ax = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ has infinitely many solutions
- ○ A is invertible
- ● A has linearly independent rows

d) Find a linear relation among the columns of A:

$$\boxed{3} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \boxed{1} \begin{pmatrix} -3 \\ -9 \\ -6 \end{pmatrix} + \boxed{0} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \boxed{0} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 0.$$

Problem 3.

[15 points]

Consider the subspace

$$V = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & 3 & 1 & 5 \end{pmatrix}.$$

a) Find a basis for $V^\perp$.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 5 \end{pmatrix} \right\}$$

(The rows of the matrix are not scalar multiples.)

b) $\dim(V) = \boxed{2}$ and $\dim(V^\perp) = \boxed{2}$.

c) Which of the following sets form a basis for V? Fill in the bubbles of all that apply.

○ $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ ● $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

○ $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -2 \\ 2 \end{pmatrix} \right\}$

● $\left\{ \begin{pmatrix} 9 \\ -3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

Problem 4. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix}.$$

a) Compute A^{-1}.

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$$

b) If $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ 2b_1 + b_2 \\ 7b_1 + 2b_2 + b_3 \end{pmatrix}$
(give a formula in terms of b_1, b_2, b_3).

(Multiply by A^{-1}.)

c) (Unrelated to the previous parts.) Use matrix algebra to justify the identity

$$(B^{-1})^T = (B^T)^{-1},$$

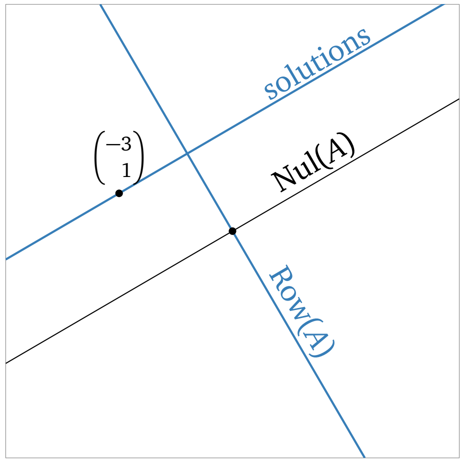
which holds for any invertible matrix B.

$$(B^{-1})^T B^T = (BB^{-1})^T = I_n^T = I_n$$

Problem 5.

[15 points]

The null space of a certain 3×2 matrix A is drawn below.



a) Draw Row(A).

b) Draw the solution set of $Ax = \begin{pmatrix}-2\\1\\0\end{pmatrix}$, given that $A\begin{pmatrix}-3\\1\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix}$.

c) Col(A) is a (○ point / ● line / ○ plane / ○ space) in $\mathbf{R}^{\boxed{3}}$.

Problem 6. [15 points]

a) For a certain 3×3 matrix A, the solution set of $Ax = (1,2,1)$ is

$$\begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix} + \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Which of the following sets are *necessarily* equal to the solution set of $Ax = b$ for *some* vector $b \in \mathbf{R}^3$? Fill in the bubbles of all that apply.

○ $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$ ● $\text{Span}\left\{\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\}$

● {} (no solutions) ● $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\}$

b) Which of the following subsets are subspaces of $\mathbf{R}^4$? Fill in the bubbles of all that apply.

○ $\text{Col}\begin{pmatrix} 1 & 7 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$ ● $\text{Span}\left\{\begin{pmatrix} 1 \\ 7 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -7 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix}\right\}$

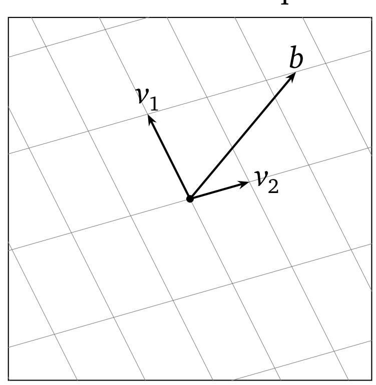
● $\text{Nul}\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \\ 7 & 0 & 2 \end{pmatrix}^T$ ○ the solution set of $\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 0 \\ 2x_1 - x_2 - x_3 = 1 \end{cases}$

○ the orthogonal complement of a plane in $\mathbf{R}^6$

c) Vectors v_1, v_2, and b are drawn below. Solve the vector equation $x_1v_1 + x_2v_2 = b$.

$x_1 =$ 1

$x_2 =$ 5/2



Problem 7.

[15 points]

In each part, provide an example or explain why no example exists.

a) A 2×2 matrix A such that $Ax = \binom{0}{0}$ is inconsistent.

No such matrix exists: $Ax = 0$ always has the solution $x = 0$.

b) A 2×2 matrix A such that $\mathrm{Col}(A) \neq \mathrm{Col}(U)$, where U is the reduced row echelon form of A.

Any matrix of rank 1 with a nonzero second row works. For instance,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow U = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

c) A 2×2 matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A^T)$.

No such matrix exists because $\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$.

d) A 2×2 matrix A with linearly *in*dependent columns and linearly *de*pendent rows.

No such matrix exists: if A has linearly independent columns then it is invertible, so its rows are also linearly independent.