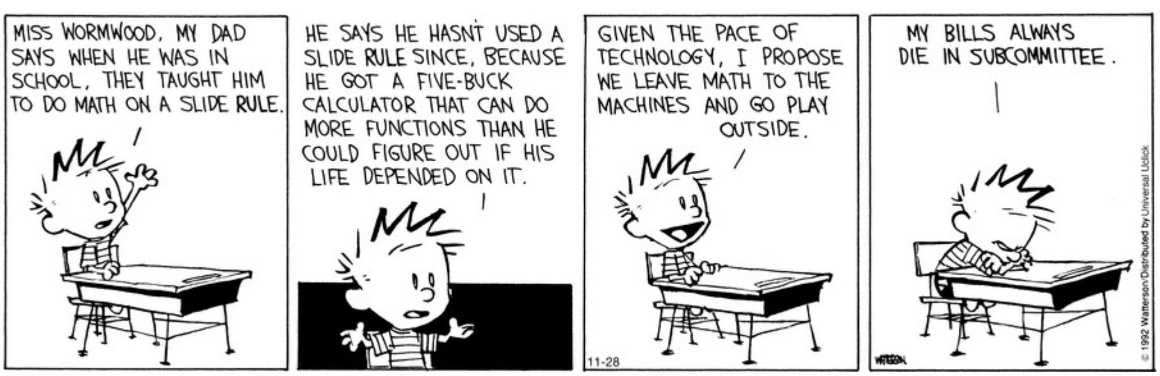
MATH 218D-1
MIDTERM EXAMINATION 1

Name		Duke Email	

usually `first.last@duke.edu`

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1.

[15 points]

a) Find a basis for the null space of the matrix

$$A = \begin{pmatrix} 2 & 6 & 1 & 1 \\ -4 & -12 & -1 & -3 \\ -2 & -6 & 0 & -2 \end{pmatrix}.$$

There are many answers. The standard procedure involving parametric vector form produces:

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

b) Which of the following statements are true about A? Fill in the bubbles of all that apply.

○ $\mathrm{Col}(A) = \mathbf{R}^3$

○ $\mathrm{Row}(A) = \mathbf{R}^4$

● $Ax = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is consistent

○ There is some $b \in \mathbf{R}^3$ such that $Ax = b$ has exactly one solution

● There is some $b \in \mathbf{R}^3$ such that $Ax = b$ is inconsistent

○ $\mathrm{Nul}(A^T) = \{0\}$

● $\begin{pmatrix} -5 \\ 1 \\ 2 \\ 2 \end{pmatrix} \in \mathrm{Nul}(A)$

○ A has linearly independent columns

Problem 2. [20 points]

Consider the subspace

$$V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -1 \end{pmatrix}.$$

a) Find a basis for $V^\perp$.

The rows are linearly indepedent, so no elimination is necessary.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

b) $\dim(V) = \boxed{2}$ and $\dim(V^\perp) = \boxed{2}$.

c) Compute the orthogonal projection b_V of the vector b onto V, where

$$b = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix}. \qquad b_V = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 4 \end{pmatrix}$$

d) The distance from b to V is $\boxed{\sqrt{6}}$.

e) Find a different vector b' <u>not in V</u> such that $b_V = b'_V$.

Add any vector in $V^\perp$ to b (or to b_V). For example,

$$b' = \begin{pmatrix} 0 \\ 5 \\ -2 \\ 4 \end{pmatrix}$$

Problem 3.

[15 points]

Consider the following matrix A and its LU decomposition:

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -4 & -1 & 3 \\ -2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{pmatrix} = LU.$$

a) Solve the matrix equation

$$Ax = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

without doing elimination.

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

b) Express L as a product of elementary matrices.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

c) Compute L^{-1}.

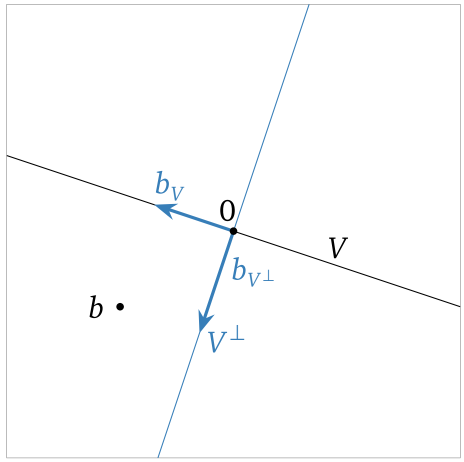
Either use the procedure for inverting a matrix using elimination, or multiply the opposite elementary matrices from (b) together in the reverse order.

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Problem 4.

[15 points]

A subspace V and a vector b are drawn below.



a) Draw the orthogonal complement $V^\perp$.

b) Draw the orthogonal projection b_V.

c) Draw the orthogonal projection $b_{V^\perp}$.
You may have drawn $b_{V^\perp}$ with the tail starting at the head of b.

Problem 5. [15 points]

Short-answer questions: no explanation is required.

a) Which of the following sets form a basis for $\mathbf{R}^3$? Fill in the bubbles of all that apply.

● $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ ● $\left\{\begin{pmatrix}20\\37\\-90\end{pmatrix}, \begin{pmatrix}0\\114\\-31\end{pmatrix}, \begin{pmatrix}0\\0\\0.15\end{pmatrix}\right\}$

○ $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}13\\24\\-193\end{pmatrix}, \begin{pmatrix}27\\-11\\22\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

○ $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}, \begin{pmatrix}10\\11\\12\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

b) Let V be a subspace of $\mathbf{R}^n$ and $b \in \mathbf{R}^n$. Suppose that $b_{V^\perp} = 0$. Which of the following can you conclude? Fill in the bubbles of all that apply.

● $b \in V$ ○ $b \in V^\perp$ ● $b = b_V$ ○ $b_V = 0$ ○ $b = b_{V^\perp}$

c) Let A be a 3×4 matrix. Suppose that the solution set of

$$Ax = \begin{pmatrix}1\\3\\2\end{pmatrix} \quad \text{is the line} \quad \begin{pmatrix}1\\7\\0\\0\end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}.$$

Which of the following sets are *definitely not* equal to the solution set of $Ax = b$ for a vector $b \in \mathbf{R}^3$? Fill in the bubbles of all that apply.

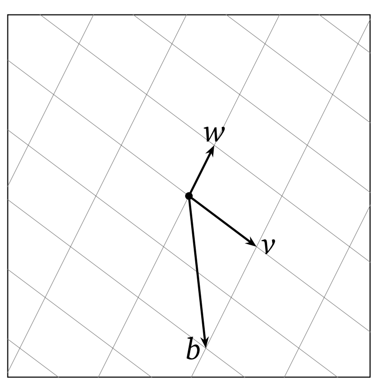
● $\{\}$ ○ $\operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}$ ○ $\begin{pmatrix}2\\3\\0\\1\end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}$ ● $\begin{pmatrix}2\\3\\0\\1\end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\right\}$

○ $\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\0\end{pmatrix}\right\}$ ● $\begin{pmatrix}2\\3\\0\\1\end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\right\}$

d) A certain 2×2 matrix

$$A = \begin{pmatrix}| & | \\ v & w \\ | & |\end{pmatrix}$$

has columns v and w, pictured here. Solve the equation $Ax = b$, where b is the vector in the picture. *You do not have enough information to compute A.*



$$x = \begin{pmatrix}1\\-2\end{pmatrix}$$

Problem 6.

[20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A 2×2 matrix A such $Ax = \binom{1}{2}$ has exactly one solution and $Ax = \binom{2}{1}$ has no solutions.
This is impossible. If $Ax = \binom{1}{2}$ has exactly one solution then A has no free columns, which means it is invertible, so it has full column rank.

b) A nonzero 2×2 matrix A such that the orthogonal projection of $b = \binom{1}{2}$ onto $\mathrm{Col}(A)$ is not equal to b.
Any matrix such that $\binom{1}{2} \notin \mathrm{Col}(A)$ works. For example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

c) A 3×2 matrix with full column rank.
This just means the columns aren't scalar multiples. For instance,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

d) A 3×3 matrix A such that the vector $(1, 1, 1)$ is contained in $\mathrm{Nul}(A)$ as well as $\mathrm{Row}(A)$.
This is impossible: $\mathrm{Nul}(A)$ and $\mathrm{Row}(A)$ are orthogonal complements of each other, but $(1, 1, 1) \cdot (1, 1, 1) \neq 0$. (This was a homework problem, in a slightly different form.)

e) A linearly independent set of three vectors in $\mathbf{R}^2$.
This is impossible: there is no 3-dimensional subspace of a plane, so any 3 vectors in $\mathbf{R}^2$ have to be linearly dependent.