

MATH 218D-1

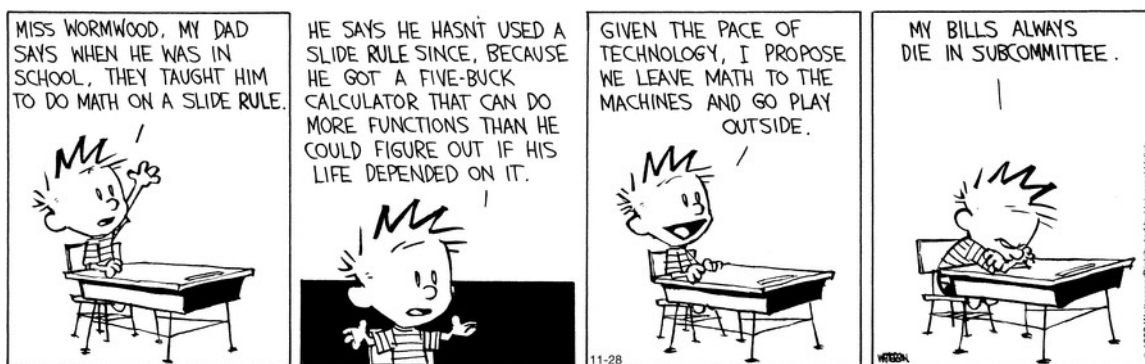
MIDTERM EXAMINATION 1

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a **3 × 5-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



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Problem 1.

[15 points]

a) Find a basis for the null space of the matrix

$$A = \begin{pmatrix} 2 & 6 & 1 & 1 \\ -4 & -12 & -1 & -3 \\ -2 & -6 & 0 & -2 \end{pmatrix}.$$

$$\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right\}$$

b) Which of the following statements are true about A ? Fill in the bubbles of all that apply.

☐ $\text{Col}(A) = \mathbf{R}^3$ ☐ $\text{Row}(A) = \mathbf{R}^4$ ☐ $Ax = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is consistent

☐ There is some $b \in \mathbf{R}^3$ such that $Ax = b$ has exactly one solution

☐ There is some $b \in \mathbf{R}^3$ such that $Ax = b$ is inconsistent

☐ $\text{Nul}(A^T) = \{0\}$ ☐ $\begin{pmatrix} -5 \\ 1 \\ 2 \\ 2 \end{pmatrix} \in \text{Nul}(A)$ ☐ A has linearly independent columns

[Scratch work for Problem 1]

Problem 2.

[20 points]

Consider the subspace

$$V = \text{Nul} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -1 \end{pmatrix}.$$

a) Find a basis for V^\perp .

$$\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right\}$$

b) $\dim(V) = \boxed{}$ and $\dim(V^\perp) = \boxed{}$.

c) Compute the orthogonal projection b_V of the vector b onto V , where

$$b = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 4 \end{pmatrix}.$$

$$b_V = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

d) The distance from b to V is $\boxed{}$.

e) Find a different vector b' not in V such that $b_V = b'_V$.

$$b' = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 2]

Problem 3.

[15 points]

Consider the following matrix A and its LU decomposition:

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -4 & -1 & 3 \\ -2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{pmatrix} = LU.$$

a) Solve the matrix equation

$$Ax = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

without doing elimination.

$$x = \begin{pmatrix} \\ \\ \end{pmatrix}$$

b) Express L as a product of elementary matrices.

$$L =$$

c) Compute L^{-1} .

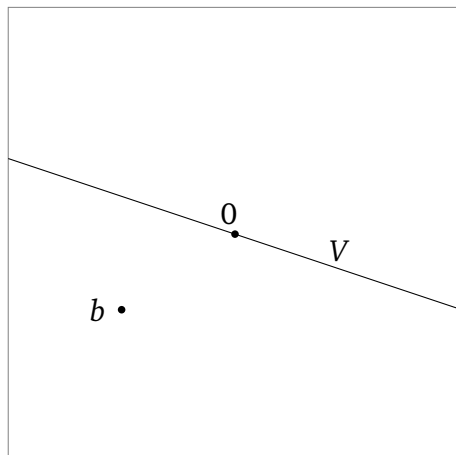
$$L^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

[Scratch work for Problem 3]

Problem 4.

[15 points]

A subspace V and a vector b are drawn below.



- a) Draw the orthogonal complement V^\perp .
- b) Draw the orthogonal projection b_V .
- c) Draw the orthogonal projection b_{V^\perp} .

[Scratch work for Problem 4]

Problem 5.

[15 points]

Short-answer questions: no explanation is required.

a) Which of the following sets form a basis for \mathbf{R}^3 ? Fill in the bubbles of all that apply.

- ☐ $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$
☐ $\left\{ \begin{pmatrix} 20 \\ 37 \\ -90 \end{pmatrix}, \begin{pmatrix} 0 \\ 114 \\ -31 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0.15 \end{pmatrix} \right\}$
- ☐ $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$
☐ $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$
☐ $\left\{ \begin{pmatrix} 13 \\ 24 \\ -193 \end{pmatrix}, \begin{pmatrix} 27 \\ -11 \\ 22 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$
- ☐ $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \right\}$
☐ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

b) Let V be a subspace of \mathbf{R}^n and $b \in \mathbf{R}^n$. Suppose that $b_{V^\perp} = 0$. Which of the following can you conclude? Fill in the bubbles of all that apply.

- ☐ $b \in V$
☐ $b \in V^\perp$
☐ $b = b_V$
☐ $b_V = 0$
☐ $b = b_{V^\perp}$

c) Let A be a 3×4 matrix. Suppose that the solution set of

$$Ax = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \text{is the line} \quad \begin{pmatrix} 1 \\ 7 \\ 0 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

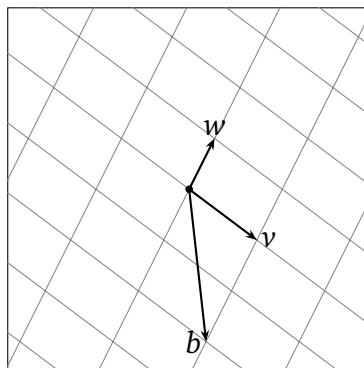
Which of the following sets are *definitely not* equal to the solution set of $Ax = b$ for a vector $b \in \mathbf{R}^3$? Fill in the bubbles of all that apply.

- ☐ $\{ \}$
☐ $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$
☐ $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$
☐ $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$
- ☐ $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$
☐ $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

d) A certain 2×2 matrix

$$A = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix}$$

has columns v and w , pictured here. Solve the equation $Ax = b$, where b is the vector in the picture. *You do not have enough information to compute A .*



$$x = \begin{pmatrix} \\ \end{pmatrix}$$

[Scratch work for Problem 5]

Problem 6.

[20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A 2×2 matrix A such $Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has exactly one solution and $Ax = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has no solutions.

b) A nonzero 2×2 matrix A such that the orthogonal projection of $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ onto $\text{Col}(A)$ is not equal to b .

c) A 3×2 matrix with full column rank.

d) A 3×3 matrix A such that the vector $(1, 1, 1)$ is contained in $\text{Nul}(A)$ as well as $\text{Row}(A)$.

e) A linearly independent set of three vectors in \mathbf{R}^2 .

[Scratch work for Problem 6]