MATH 218D-1
PRACTICE MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1. [10 points]

a) Compute the $A = QR$ decomposition of the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 7 & -7 \\ 2 & 2 & 3 \end{pmatrix}.$$

$$Q = \begin{pmatrix} 1/\sqrt{14} & 1/\sqrt{6} & 4/\sqrt{21} \\ 3/\sqrt{14} & 1/\sqrt{6} & -2/\sqrt{21} \\ 2/\sqrt{14} & -2/\sqrt{6} & 1/\sqrt{21} \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{14} & 2\sqrt{14} & -\sqrt{14} \\ 0 & \sqrt{6} & -2\sqrt{6} \\ 0 & 0 & \sqrt{21} \end{pmatrix}$$

b) Briefly explain how a QR decomposition makes it possible to quickly compute the least-squares solution of $Ax = b$.

Solving $A^T A\widehat{x} = A^T b$ is the same as solving $R\widehat{x} = Q^T b$, which only requires substitution because R is in row echelon form.

Problem 2. [20 points]

Consider the subspace

$$V = \text{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\right\}.$$

a) Find a basis for $V^\perp$.

$$\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}$$

We have to compute a basis for $\text{Nul}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 0 & -1 & 1\end{pmatrix}$.

b) Compute the projection matrix P_V.

$$P_V = \frac{1}{11}\begin{pmatrix}5 & 2 & -1 & 5\\ 2 & 3 & 4 & 2\\ -1 & 4 & 9 & -1\\ 5 & 2 & -1 & 5\end{pmatrix}$$

The easiest thing is to use the Horrible Formula.

c) Find the orthogonal projection of the vector $b = (1, 3, 0, 0)$ onto V.

$$b_V = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Just multiply b by your answer to **b)**.

d) Find an invertible matrix C and a diagonal matrix D such that $P_V = CDC^{-1}$.

$$C = \begin{pmatrix}1 & 1 & 1 & -1\\ 1 & 0 & -2 & 0\\ 1 & -1 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} \qquad D = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

You're given a basis for V (the 1-eigenspace), and you computed a basis for $V^\perp$ (the 0-eigenspace) in **a)**.

Problem 3. [10 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & -6 & -18 \\ -2 & 3 & 14 \\ 1 & -1 & -6 \end{pmatrix}.$$

a) Compute the characteristic polynomial of A.

$$p(\lambda) = -\lambda^3 - 2\lambda^2 + \lambda + 2$$

b) Which of the following numbers is an eigenvalue of A?

○ -4 ● -2 ○ 0 ○ 2 ○ 4

Substitute each value into $p(\lambda)$ and check if you get zero.

c) Find any eigenvector of A.

$$v = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Find a vector in $\mathrm{Nul}(A + 2I_3)$.

Problem 4. [10 points]

Consider the difference equation

$$v_k = A^k v_0 \qquad A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \qquad v_0 = \begin{pmatrix} -4 \\ -3 \end{pmatrix}.$$

a) Find an expression for v_k that does not involve matrix multiplication.

First we compute the characteristic polynomial:

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

Then we compute eigenvectors:

$$w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Now we expand in the eigenbasis:

$$v_0 = x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \Longrightarrow \quad x_1 = -1,\ x_2 = -1.$$

Finally, we multiply by A^k:

$$v_k = A^k v_0 = -\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 2^k \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

b) If $v_k = \binom{x_k}{y_k}$, the ratio x_k / y_k approaches $\boxed{1}$ as $k \to \infty$.

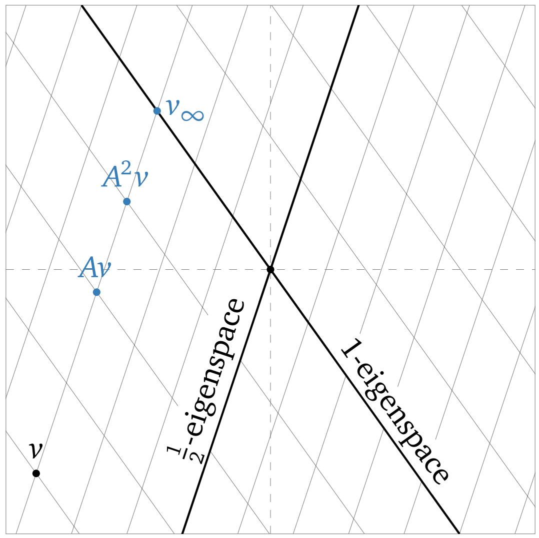
Problem 5.

[10 points]

The eigenspaces of a certain matrix A are drawn below.

a) A vector v is drawn as a point in the plane. Draw *and label* the vectors Av and A^2v as points in the plane.

b) Draw *and label* $v_\infty = \lim_{k\to\infty} A^k v$ as a point in the plane.



Problem 6.

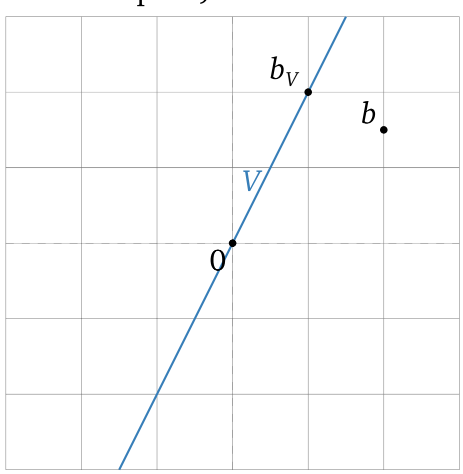
[10 points]

For a certain subspace V of $\mathbf{R}^2$, a vector b and its orthogonal projection b_V are drawn below.

a) Draw the subspace V.

b) Find the projection matrix P_V.

(Grid marks are one unit apart.)



$$P_V = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Problem 7. [15 points]

Short-answer questions: no justification is necessary.

a) A certain plane V in $\mathbf{R}^3$ has projection matrix

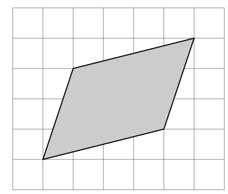
$$P_V = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Find **i)** a basis for V and **ii)** a basis for $V^\perp$.

$$V: \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\} \qquad V^\perp: \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

For V, take any two columns of P_V. For $V^\perp$, take the cross product of those two vectors. (I divided my answer by -3.)

b) Compute the area of the paralellogram in the picture.
(Grid marks are one unit apart.)



area = 11

c) A certain $n \times n$ matrix has characteristic polynomial

$$p(\lambda) = \lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24 = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4).$$

Which of the following can you determine from this information?

- ● The number n.
- ● The trace of A.
- ● The determinant of A.
- ● The eigenvalues of A.
- ○ The eigenvectors of A.
- ● Whether A is invertible.
- ● Whether A is diagonalizable.

d) If A is a 2×2 matrix that is neither diagonalizable nor invertible, then its characteristic polynomial is $p(\lambda) =$ λ^2.

Problem 8. [15 points]

In each part, provide an example or explain why no example exists. All matrices have *real entries.*

a) A non-diagonalizable 2×2 matrix with characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$.

No such matrix exists: it has two distinct eigenvalues, 0 and 1.

b) A non-invertible 2×2 matrix with a complex (non-real) eigenvalue.

No such matrix exists: it has two distinct eigenvalues, λ and $\overline{\lambda}$, so its determinant is $|\lambda|^2 \neq 0$.

c) A matrix Q with *orthonormal columns* such that $\operatorname{rank}(QQ^T) = 2$.

Any matrix with two orthonormal columns will do, since QQ^T is the projection matrix onto $\operatorname{Col}(A)$. For example,

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

d) A nonzero matrix A such that $b_V = 0$, where $b = (1, 1, 1)$ and $V = \operatorname{Col}(A)$.

This means that $b \in \operatorname{Col}(A)^\perp$, so we just have to find some vectors orthogonal to b. For instance,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$