MATH 218D-1
MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

Problem 1. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & -3 & 0 \end{pmatrix}.$$

a) Which of the following is an eigenvector of A? What is the eigenvalue?

○ $\begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ ● $\begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\lambda = \boxed{0}$

Multiply A by each vector. Note that if $Av = 0$ then $Av = 0v$.

b) Find a basis for the 1-eigenspace of A.

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

You have to row reduce $A - I_3$.

c) The characteristic polynomial of A is

$$p(\lambda) = \lambda(\lambda - 1)^2(\lambda + 1).$$

Given this and what you know about A from parts **a)** and **b)**, can you determine if A is diagonalizable?

● yes ○ no ○ not enough information

Explain.

We see that the algebraic multiplicities of 0 and −1 are both 1, and we computed the geometric multiplicity of 1 to be 2 in **b)**, so this matrix satisfies the AM/GM criterion.

Problem 2. [15 points]

A certain matrix A has eigenvectors

$$\begin{pmatrix}1\\2\\0\end{pmatrix} \quad \begin{pmatrix}-1\\2\\1\end{pmatrix} \quad \begin{pmatrix}2\\1\\-1\end{pmatrix}$$

with respective eigenvalues $1, -1$, and 0.

a) Compute. (Your answer should only have numbers in it.)

$$A^{100}\begin{pmatrix}-2\\7\\3\end{pmatrix} = \begin{pmatrix}0\\8\\2\end{pmatrix}$$

First you expand $(-2, 7, 3)$ in the eigenbasis:

$$\begin{pmatrix}-2\\7\\3\end{pmatrix} = 2\begin{pmatrix}1\\2\\0\end{pmatrix} + 2\begin{pmatrix}-1\\2\\1\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix}.$$

Multiplying by A^{100} gives:

$$A^{100}\begin{pmatrix}-2\\7\\3\end{pmatrix} = 1^{100}\cdot 2\begin{pmatrix}1\\2\\0\end{pmatrix} + (-1)^{100}\cdot 2\begin{pmatrix}-1\\2\\1\end{pmatrix} - 0^{100}\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\8\\2\end{pmatrix}.$$

b) $A = CDC^{-1}$ where

$$C = \begin{pmatrix}1 & -1 & 2\\2 & 2 & 1\\0 & 1 & -1\end{pmatrix} \qquad D = \begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{pmatrix}$$

Your'e given a diagonalization in vector form in the statement of the problem.

c) What are the eigenvalues of A^{100}? 1 and 0

The eigenvalues are λ^{100}, where λ is an eigenvalue of A.

d) Find a basis for Nul(A).

$$\left\{\begin{pmatrix}2\\1\\-1\end{pmatrix}\right\}$$

The null space is the 0-eigenspace, and you're given a basis for each eigenspace.

e) Find a basis for Col(A).

$$\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix}\right\}$$

You showed in the homework that any λ-eigenvector is in Col(A) if $\lambda \neq 0$. By rank-nullity, Col(A) is a plane, and you have two linearly independent vectors in Col(A), so they form a basis.

Problem 3.

[20 points]

Consider the subspace

$$V = \operatorname{Span}\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}2\\2\\3\end{pmatrix}\right\}.$$

a) Find an orthogonal basis for V.

$$\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\}$$

This requires one iteration of Gram–Schmidt.

b) Compute the orthogonal projection onto V of $b = \begin{pmatrix}-6\\0\\3\end{pmatrix}$.

$$b_V = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

Use the projection formula.

c) Find a basis for $V^\perp$.

$$\left\{\begin{pmatrix}-5\\-1\\4\end{pmatrix}\right\}$$

Since V is a plane, $V^\perp$ is a line, so you just need to find a nonzero vector that's orthogonal to your basis vectors. Either take the cross product of your basis vectors, or compute $b_{V^\perp} = b - b_V$.

d) Compute the projection matrix P_V.

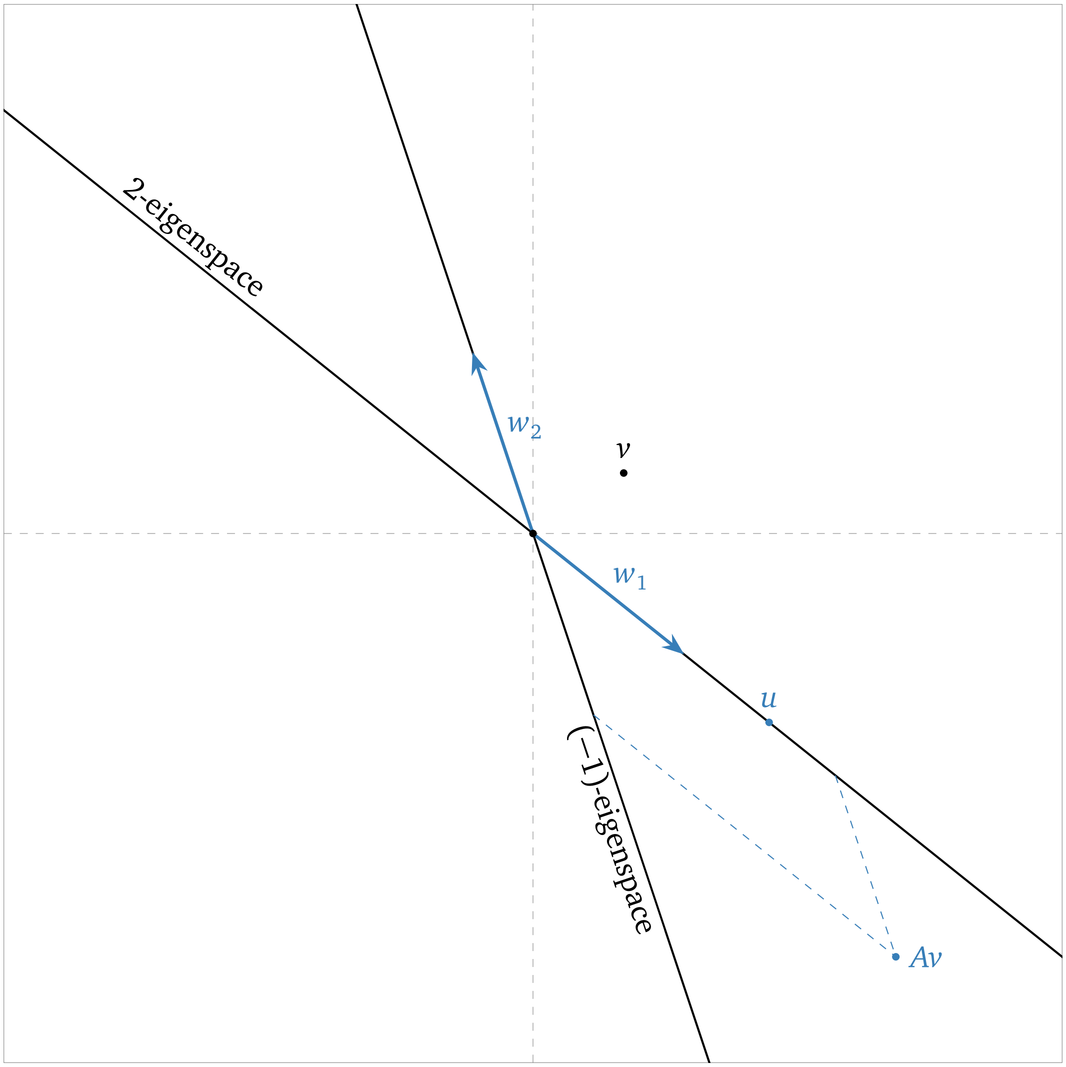
$$\frac{1}{42}\begin{pmatrix}17 & -5 & 20\\-5 & 41 & 4\\20 & 4 & 26\end{pmatrix}$$

It's easier to find $P_{V^\perp}$ using the formula for the matrix of projection onto a line, then compute $P_V = I_3 - P_{V^\perp}$.

Problem 4.

[10 points]

The eigenspaces of a certain 2×2 matrix are drawn below, along with a vector v.



a) Draw and label a 2-eigenvector w_1 and a (-1)-eigenvector w_2 such that $v = w_1 + w_2$.

b) Draw and label Av as a point in the plane.

c) Draw and label $u = \lim_{k \to \infty} \dfrac{A^k v}{||A^k v||}$ as a point in the plane. One unit is the length of this line:

Problem 5.

[20 points]

Short-answer questions: no justification is necessary unless stated otherwise.

a) Find any (real or complex) eigenvector of the matrix $\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 2 \\ i\sqrt{6} \end{pmatrix}$$

Use the quadratic formula to find the eigenvalue $1 \pm i\sqrt{6}$, then use the $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ trick.

b) In a QR decomposition $A = QR$, briefly derive the identity $R = Q^T A$ using matrix algebra.

Multiply both sides by Q^T:

$$Q^T A = Q^T QR.$$

Since $Q^T Q = I_m$, this cancels to give $Q^T A = R$.

c) For which value(s) of t, if any, do the following vectors *not* form a basis of $\mathbf{R}^4$?

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} t \\ 5 \\ 6 \\ 8 \end{pmatrix} \right\} \qquad t = \boxed{4}$$

It's easiest to compute the determinant of

$$\begin{pmatrix} 1 & 1 & 4 & t \\ 2 & 2 & 5 & 5 \\ 3 & 3 & 6 & 6 \\ 4 & 5 & 7 & 8 \end{pmatrix}$$

using row operations.

d) Suppose that the characteristic polynomial of A is $p(\lambda) = -(\lambda-1)^2(\lambda-2)$. Which of the following statements do you know to be true? Fill in the bubbles of all that apply.

- ● $\det(A) = 2$.
- ● $\det(A - 2I_3) = 0$.
- ○ $(1, 1, 2)$ is an eigenvector of A.
- ○ A is diagonalizable.
- ● All of the eigenvalues of A are real.
- ○ $A - I_3$ has two free variables.
- ● 0 is an eigenvalue of $A - I_3$.
- ● $\mathrm{Tr}(A) = 4$.

e) The matrix $\begin{pmatrix} 2 & 1 \\ 0 & a \end{pmatrix}$ is diagonalizable unless $a = \boxed{2}$.

This matrix is upper triangular, so its eigenvalues are the diagonal entries. If it has two eigenvalues, it's automatically diagonalizable.

Problem 6. [20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.) *All matrices must have real entries.*

a) A 3×3 matrix whose 1-eigenspace is

$$\text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}.$$

You can construct $A - I_3$ by reverse-engineering parametric vector form, but it's probably easier to just compute the projection matrix for this line:

$$P_V = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix}.$$

b) A 2×2 matrix with no real eigenvalues.

We had this example in class:

$$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}.$$

Or you could use the matrix from Problem 5(a).

c) A 3×2 matrix Q with orthonormal columns such that $QQ^T = I_3$.

No such matrix exists: QQ^T is the projection matrix onto the column space of Q, which is a *plane*; hence this projection matrix is not the identity.

d) A *unit* vector $v \in \mathbf{R}^3$ such that

$$\left\| v \times \begin{pmatrix}1\\2\\2\end{pmatrix} \right\| = 4$$

No such vector exists. We have

$$\left\| v \times \begin{pmatrix}1\\2\\2\end{pmatrix} \right\| = \|v\| \cdot \left\| \begin{pmatrix}1\\2\\2\end{pmatrix} \right\| \sin(\theta) = 3\sin(\theta) < 4,$$

where θ it the angle from v to $(1, 2, 2)$.

e) An *upper-triangular* 3×3 matrix with a complex (non-real) eigenvalue.

No such matrix exists. The eigenvalues of an upper-triangular matrix are the diagonal entries, and all matrices in the problem have real entries.