MATH 218D-1
MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

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Problem 1. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & -3 & 0 \end{pmatrix}.$$

a) Which of the following is an eigenvector of A? What is the eigenvalue?

○ $\begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\lambda = \boxed{}$

b) Find a basis for the 1-eigenspace of A.

$$\left\{ \qquad \qquad \right\}$$

c) The characteristic polynomial of A is

$$p(\lambda) = \lambda(\lambda - 1)^2(\lambda + 1).$$

Given this and what you know about A from parts **a)** and **b)**, can you determine if A is diagonalizable?

○ yes ○ no ○ not enough information

Explain.

[Scratch work for Problem 1]

Problem 2.

[15 points]

A certain matrix A has eigenvectors

$$\begin{pmatrix}1\\2\\0\end{pmatrix} \quad \begin{pmatrix}-1\\2\\1\end{pmatrix} \quad \begin{pmatrix}2\\1\\-1\end{pmatrix}$$

with respective eigenvalues $1, -1$, and 0.

a) Compute. (Your answer should only have numbers in it.)

$$A^{100}\begin{pmatrix}-2\\7\\3\end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

b) $A = CDC^{-1}$ where

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

c) What are the eigenvalues of A^{100}?

d) Find a basis for Nul(A).

$$\left\{ \quad \right\}$$

e) Find a basis for Col(A).

$$\left\{ \quad \right\}$$

[Scratch work for Problem 2]

Problem 3. [20 points]

Consider the subspace

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

a) Find an orthogonal basis for V.

$$\left\{ \qquad \right\}$$

b) Compute the orthogonal projection onto V of $b = \begin{pmatrix} -6 \\ 0 \\ 3 \end{pmatrix}$.

$$b_V = \begin{pmatrix} \quad \end{pmatrix}$$

c) Find a basis for $V^{\perp}$.

$$\left\{ \qquad \right\}$$

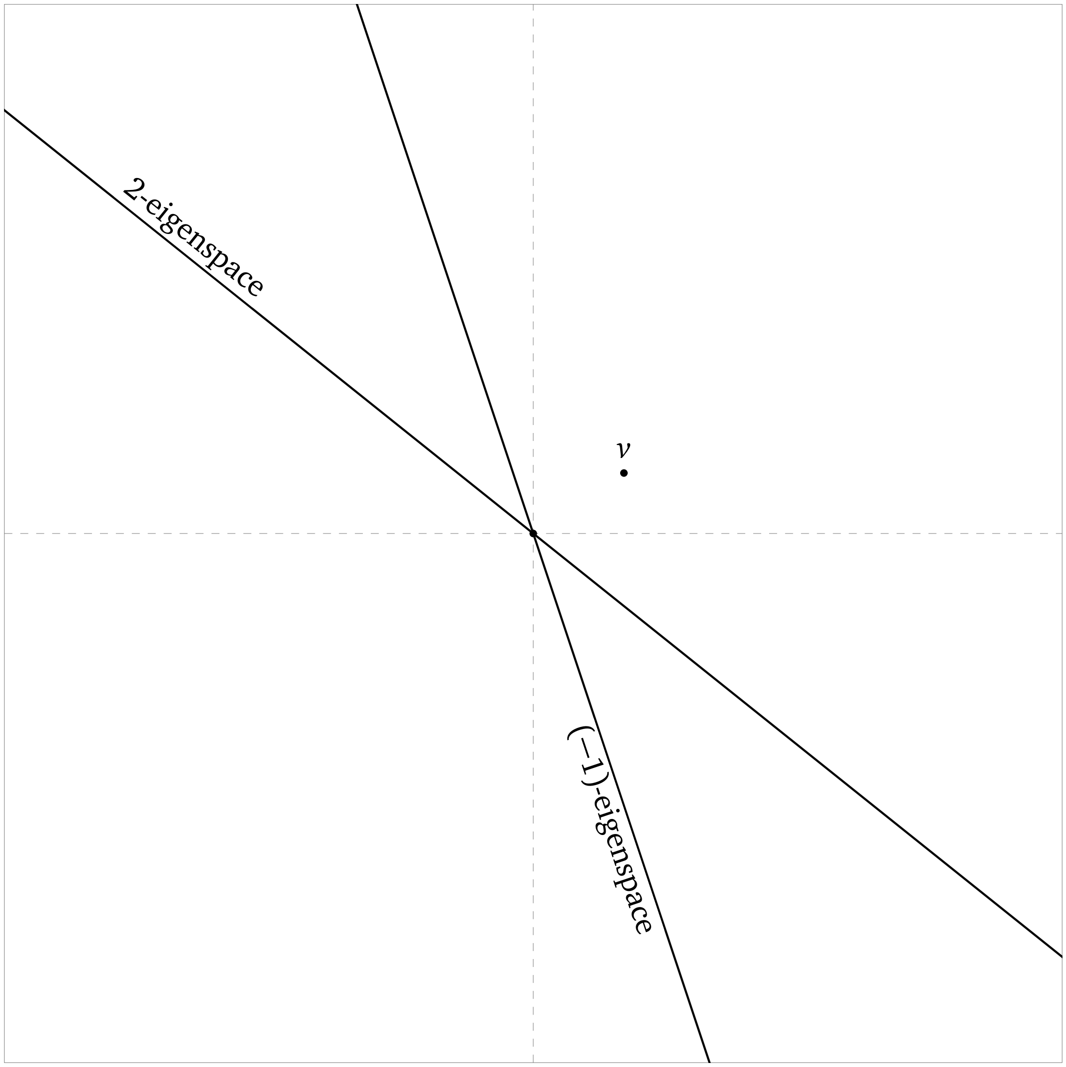
d) Compute the projection matrix P_V.

$$\begin{pmatrix} \quad \end{pmatrix}$$

[Scratch work for Problem 3]

Problem 4. [10 points]

The eigenspaces of a certain 2×2 matrix are drawn below, along with a vector v.



a) Draw and label a 2-eigenvector w_1 and a (-1)-eigenvector w_2 such that $v = w_1 + w_2$.

b) Draw and label Av as a point in the plane.

c) Draw and label $u = \lim_{k\to\infty} \dfrac{A^k v}{||A^k v||}$ as a point in the plane. One unit is the length of this line:

____

[Scratch work for Problem 4]

Problem 5.

[20 points]

Short-answer questions: no justification is necessary unless stated otherwise.

a) Find any (real or complex) eigenvector of the matrix $\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$.

$$\begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

b) In a QR decomposition $A = QR$, briefly derive the identity $R = Q^T A$ using matrix algebra.

c) For which value(s) of t, if any, do the following vectors *not* form a basis of $\mathbf{R}^4$?

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} t \\ 5 \\ 6 \\ 8 \end{pmatrix} \right\} \qquad t = \boxed{}$$

d) Suppose that the characteristic polynomial of A is $p(\lambda) = -(\lambda - 1)^2(\lambda - 2)$. Which of the following statements do you know to be true? Fill in the bubbles of all that apply.

- ○ $\det(A) = 2$.
- ○ $\det(A - 2I_3) = 0$.
- ○ $(1, 1, 2)$ is an eigenvector of A.
- ○ A is diagonalizable.
- ○ All of the eigenvalues of A are real.
- ○ $A - I_3$ has two free variables.
- ○ 0 is an eigenvalue of $A - I_3$.
- ○ $\mathrm{Tr}(A) = 4$.

e) The matrix $\begin{pmatrix} 2 & 1 \\ 0 & a \end{pmatrix}$ is diagonalizable unless $a = \boxed{}$.

[Scratch work for Problem 5]

Problem 6.

[20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.) *All matrices must have real entries.*

a) A 3×3 matrix whose 1-eigenspace is

$$\operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}.$$

b) A 2×2 matrix with no real eigenvalues.

c) A 3×2 matrix Q with orthonormal columns such that $QQ^T = I_3$.

d) A *unit* vector $v \in \mathbf{R}^3$ such that

$$\left\| v \times \begin{pmatrix}1\\2\\2\end{pmatrix} \right\| = 4$$

e) An *upper-triangular* 3×3 matrix with a complex (non-real) eigenvalue.

[Scratch work for Problem 6]