Math 218D-1: Homework #1

due Wednesday, September 3, at 11:59pm

> The homework assignments in this class will require *a lot of work*. You should expect to spend around 10 hours per week solving them—more for some people, less for others—so allocate your time accordingly. There are good reasons for this: on the one hand, I have to make sure that you've had practice with any concept or computation that you'll see on an exam, but more importantly, it simply takes time to get good at math (or anything else).
>
> This first homework is relatively light on concepts. The homework will become heavily conceptual starting around week 3.

1. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u+v+w$ and $u+2v-w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Express $u+3v-w$ as a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to _____?

e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

2. Express the vector $(-1, 2, 3)$ as a linear combination of the unit coordinate vectors e_1, e_2, e_3.
(See the notes for the definition of the standing notation "e_i".)

3. Decide if each statement is true or false, and explain why. In this problem, v and w denote vectors in $\mathbf{R}^n$.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) The vector 0 is a linear combination of v and w.

c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

d) If $v \neq 0$ then $v \cdot v > 0$.

e) $v^T w = v \cdot w$.

4. Let v and w be vectors in $\mathbf{R}^n$. Suppose that $v \cdot v = 1$, $w \cdot w = 2$, and $v \cdot w = 3$. Compute the following quantities using the algebra of dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v+w)\cdot(v-w)$ **c)** $(v+2w)\cdot(3v)$.

5. **a)** Find a nonzero vector $v \in \mathbf{R}^3$ such that $v \cdot (1,1,1) = 0$.

b) Find a nonzero vector $w \in \mathbf{R}^3$ such that $w \cdot (1,1,1) = 0$ and $w \cdot v = 0$, where v is your vector from **a)**.

6. Compute the following matrix-vector products using *both* the by-row and by-column methods. If the product is not defined, explain why. *Show your work.*

a) $\begin{pmatrix} 2 \\ 5 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & -2 \\ 0 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ **c)** $\begin{pmatrix} 7 & 2 & 4 \\ 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 7 & 4 \\ -2 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ **e)** $\begin{pmatrix} 2 & 6 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}$

Check your work using SymPy on the Sage cell on the website, as in:

```
A = Matrix([[7,  2, 4],
            [3, -3, 1]])
  # Shortcut for specifying a column vector:
x = Matrix([1, -1, 1])
pprint(A*x)
```

7. For each pair of vectors u and v, compute $u^T v$ and $u \cdot v$.

a) $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ **b)** $u = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix}, v = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix}$

Check your work using SymPy on the Sage cell on the website, as in:

```
  # Shortcut for specifying a column vector:
u = Matrix([1, 2, 3])
v = Matrix([4, 5, 6])
pprint(u.T*v)
pprint(u.dot(v))
```

8. Suppose that $u = (a,b,c)$ and $v = (d,e,f)$ are vectors satisfying $2u + 3v = 0$. *Without doing any computations at all,* find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

9. Let A be a 4×3 matrix satisfying

$$Ae_1 = \begin{pmatrix}1\\3\\2\\7\end{pmatrix} \qquad Ae_2 = \begin{pmatrix}4\\4\\-1\\-1\end{pmatrix} \qquad Ae_3 = \begin{pmatrix}0\\0\\-2\\1\end{pmatrix}.$$

Find A, *without doing any computations at all.*

10. Find the matrix A satisfying

$$A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x+3y\\-x-2y\end{pmatrix}.$$

[**Hint:** compute $A\binom{1}{0}$ and $A\binom{0}{1}$.]

11. Suppose that A is a 4×3 matrix such that

$$A\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}7\\1\\2\\9\end{pmatrix} \qquad A\begin{pmatrix}0\\1\\4\end{pmatrix} = \begin{pmatrix}1\\-4\\2\\3\end{pmatrix}.$$

Let x be the vector

$$x = \begin{pmatrix}3\\-2\\-2\end{pmatrix} = 3\begin{pmatrix}1\\0\\2\end{pmatrix} - 2\begin{pmatrix}0\\1\\4\end{pmatrix}.$$

Find Ax *without trying to solve for* A.

12. Suppose that A is a 2×3 matrix and that B and C are 2×2 matrices. Which of the following expressions are defined, and which are undefined?

a) $2A$ **b)** $2A-B$ **c)** $2B-C$

d) AB **e)** BA **f)** A^TB

g) B^2 **h)** $BA-A$ **i)** $(B-I_2)A$

13. Suppose that A, B, and C are 3×3 matrices. Simplify the following expressions (write them without parentheses or identity matrices):

a) $(A+B)^2$ **b)** $A(2I_3-B)C$ **c)** $(AB)^TC$

d) $C(A+3B)^T$ **e)** $(A^T+I_3)^TC$ **f)** $(A^TA)^T$

14. Compute the matrix products

$$\textbf{a)}\ \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

in two ways: using the column form, and using the outer product form. *Show your work.*

Check your work using SymPy on the Sage cell on the website, as in:

```
A = Matrix([[1,  2],
            [2, -1]])
B = Matrix([[2,  1, -1],
            [4, -1,  2]])
pprint(A*B)
```

15. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \qquad C = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Note that B and C are diagonal.

a) Compute AB and BA.

b) How do the columns or rows of a matrix change when it is multiplied by a diagonal matrix on the right or left?

c) Compute BC and CB.

d) What happens when you multiply two diagonal matrices together?

16. Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

a) For which 2×2 matrices B does $AB = BA$?

b) Find *nonzero* 2×2 matrices B and C such that $B \neq C$ yet $AB = AC$.

17. Recall that a matrix A is *symmetric* if $A^T = A$. Decide if each statement is true or false. If it is true, explain why; if it is false, provide a counterexample.

a) A symmetric matrix is square.

b) If A and B are symmetric of the same size, then AB is symmetric.

c) If A is symmetric, then A^3 is symmetric.

d) If A is any matrix, then A^TA is symmetric.

18. In the table below, a linear system is expressed as a system of equations, as a matrix equation, as a vector equation, or as an augmented matrix. Fill in the rows of the table with the other three equivalent ways of writing each system of equations.

System of Equations	Matrix Equation	Vector Equation	Augmented Matrix
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 \qquad\quad + 4x_3 &= 2 \end{aligned}$			
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$		
		$x_1 \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	
			$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

19. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= 2 \\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Rewrite the system as an augmented matrix.

b) Use row replacements to eliminate x_1 from the second and third equations.

c) Use a row replacement to eliminate x_2 from the third equation (with x_1 still only appearing in the first).

d) Translate your augmented matrix back into a system of equations.

e) Solve for x_3, then for x_2, then for x_1. What is the solution?

Show your work.

20. **(Internalizing a Definition)** Which of the following matrices are *not* in row echelon form? Why not?

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

21. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

$$\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$

22. **(Internalizing a Definition)** Which of the following matrices are *not* in reduced row echelon form? Why not?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

23. **(Practicing a Procedure)** Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use Rabinoff's Reliable Row Reducer, but *write out all row operations you perform.*

a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b)** $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$ **c)** $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ **d)** $\left(\begin{array}{ccc|c} 0 & 0 & 2 & 2 \\ 1 & -1 & 3 & -1 \\ -1 & 1 & 1 & 8 \end{array}\right)$

Check your work using SymPy. For instance, in **a)** you would do something like:

```
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]])
pprint(A.echelon_form())
pprint(A.rref(pivots=False))
```

(Omitting `pivots=False` would cause Sympy to print the pivot locations as well. Note that Sympy may produce a different REF than you.)

By the way, Sympy has no notion of an augmented matrix—the augmentation line only exists to help a human remember that it came from a system of equations. To solve **b)** in Sympy, you would do something like:

```
A = Matrix([[1, 1, 0, 1],
            [1, 2, 1, 1],
            [0, 1, 2, 2]])
  # or even fancier:
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]]).row_join(Matrix([1, 1, 2]))
```

24. **(Practicing a Procedure)** Solve each of the following systems of equations (they all have a unique solution).

a) $$\begin{aligned} -x_1 \quad\quad\quad + 4x_3 &= 2 \\ 3x_1 + 2x_2 + 4x_3 &= 8 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

b) $$\begin{pmatrix} 0 & 3 & 2 \\ 1 & 3 & -3 \\ 4 & 9 & -16 \end{pmatrix} x = \begin{pmatrix} 11 \\ -7 \\ -47 \end{pmatrix}$$

c) $$x_1 \begin{pmatrix} 2 \\ 6 \\ 4 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 6 \\ -4 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -5 \\ -5 \\ 5 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ -7 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 7 \\ -12 \end{pmatrix}$$

25. The parabola $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,9)$, $(-1,6)$. Find the coefficients a, b, c.