# Math 218D-1: Homework #10

due Wednesday, November 5, at 11:59pm

SymPy knows what a complex number is: write `3+4*I` for $3+4i$.

**1.** **(Practicing a Procedure)** For each of the following $2\times 2$ matrices $A$ from HW9#11, find an invertible matrix $C$ and a diagonal matrix $D$ such that $A = CDC^{-1}$. (This requires no additional work.)

**a)** $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

**2.** **(Practicing a Procedure)** For each of the following matrices $A$ from HW9#13, find an invertible matrix $C$ and a diagonal matrix $D$ such that $A = CDC^{-1}$. (This requires no additional work.)

**a)** $\begin{pmatrix} -1 & 7 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 7 & 12 & 12 \\ -8 & -13 & -12 \\ 4 & 6 & 5 \end{pmatrix}$

Once you're comfortable with this procedure, you can use SymPy:

```
A = Matrix([[ 7,  12,  12],
            [-8, -13, -12],
            [ 4,   6,   5]])
C, D = A.diagonalize()
pprint(C)
pprint(D)
```

Of course, SymPy might produce a different diagonalization than you. See Problem 3.

**3.** **(Internalizing a Concept)** Consider the matrix

$$A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

**a)** Find a diagonal matrix $D$ and an invertible matrix $C$ such that $A = CDC^{-1}$.

**b)** Find a *different* diagonal matrix $D'$ and a *different* invertible matrix $C'$ such that $A = C'D'C'^{-1}$.

[**Hint:** Try re-ordering the eigenvalues and choosing a different eigenbasis.]

**4. (Internalizing a Concept)** Compute the matrix with eigenvalues $0, 1, 2$ and corresponding eigenvectors

$$\begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\2\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\\2\end{pmatrix}.$$

(There is only one such matrix.)

**5.** Let $A$ and $B$ be $n \times n$ matrices, and let $v_1, \ldots, v_n$ be a basis of $\mathbf{R}^n$.

**a)** Suppose that each $v_i$ is an eigenvector of both $A$ and $B$. Show that $AB = BA$.

**b)** Suppose that each $v_i$ is an eigenvector of both $A$ and $B$ with the same eigenvalue. Show that $A = B$.

[**Hint:** Hint: use the matrix form of diagonalization.]

**6. (Exploration Problem)** Let $A$ be an $n \times n$ matrix, and let $C$ be an invertible $n \times n$ matrix. Prove that the characteristic polynomial of $CAC^{-1}$ equals the characteristic polynomial of $A$.

[**Hint:** Distribute $C(A - I_n)C^{-1}$ and take determinants.]

In particular, $A$ and $CAC^{-1}$ have the same eigenvalues, the same determinant, and the same trace. They are called *similar* matrices.

**7. (Exploration Problem)** Let $V$ be a plane in $\mathbf{R}^3$, let $L = V^\perp$ be the orthogonal line, let $P_L$ be the matrix for orthogonal projection onto $L$, and let $R_V = I_3 - 2P_L$ be the reflection over $V$, as in HW8#8.

**a)** Prove that there exists an invertible $3 \times 3$ matrix $C$ such that

$$P_L = C \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} C^{-1}.$$

Use this and Problem 6 to show that the characteristic polynomial of $P_L$ is $-\lambda^2(\lambda - 1)$.

**b)** Use **a)** and some matrix algebra to show that

$$R_V = C \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix} C^{-1}$$

for the same matrix $C$ of part **a)**. Use this to show that the characteristic polynomial of $R_V$ is $-(\lambda - 1)^2(\lambda + 1)$ and that $\det(R_V) = -1$.

(Compare HW8#8 and HW9#12.)

**8.** **(Driving a Point Home)** Consider the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

from Problem 1(a). Find a closed formula for $A^k$: that is, an expression of the form

$$A^k = \begin{pmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{pmatrix},$$

where $a_{ij}(k)$ is a function of $k$. This lets you compute $A^k$ in constant time.
[**Hint:** This is a diagonalization problem.]

**9.** **(Internalizing a Concept)** Consider the difference equation

$$v_{k+1} = Av_k \qquad A = \begin{pmatrix} 6 & 1 & -10 \\ -4 & 1 & 10 \\ 2 & 0 & -4 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

We can diagonalize $A = CDC^{-1}$ with

$$C = \begin{pmatrix} 3 & -5 & 2 \\ -2 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} \qquad C^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 5 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**a)** Expand $v_0$ in the eigenbasis *without doing any work*.

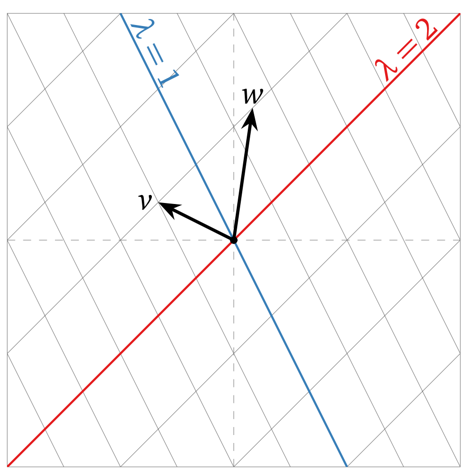
**b)** Compute $v_k = A^k v_0$.

**10.** **(Picture Problem)** A certain $2\times2$ matrix $A$ has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.

**a)** Draw $Av$, $A^2v$, and $Aw$.

**b)** Compute the limit of $A^n v / \|A^n v\|$ as $n \to \infty$.

(Recall that $A^n v / \|A^n v\|$ is the unit vector in the direction of $A^n v$.)



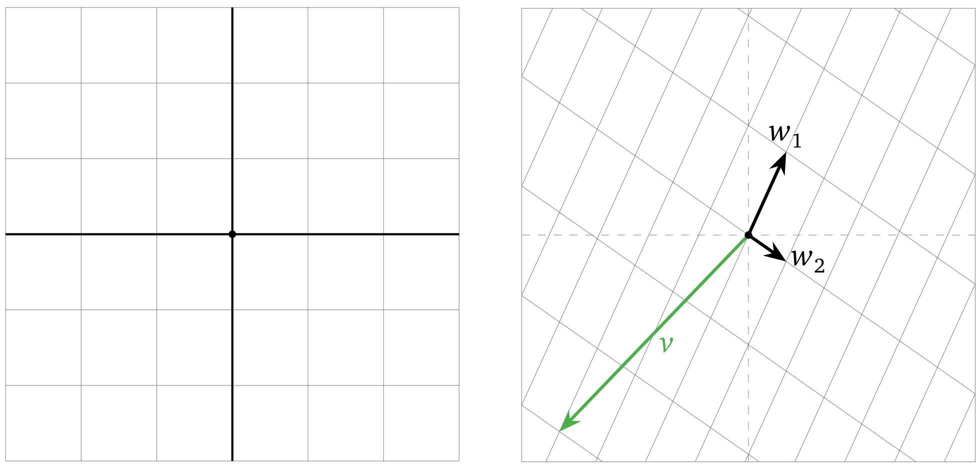
**11.** **(Picture Problem)** A certain diagonalizable $2\times 2$ matrix $A$ is equal to $CDC^{-1}$, where $C$ has columns $w_1, w_2$ pictured below, and $D = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$.

**a)** Draw $C^{-1}v$ on the left.

**b)** Draw $DC^{-1}v$ on the left.

**c)** Draw $Av = CDC^{-1}v$ on the right.

**d)** What happens to $A^n v$ as $n \to \infty$?




**12.** **(Practicing a Procedure)** Compute the following complex numbers.

**a)** $(1+i)+(2-i)$ **b)** $(1+i)(2-i)$ **c)** $\overline{2-i}$ **d)** $\dfrac{1+i}{2-i}$

**e)** $|1+i|$ **f)** $2e^{2\pi i/3}$ **g)** $5e^{3\pi i}$

**13.** **(Practicing a Procedure)** Express each complex number in polar coordinates $re^{i\theta}$.

**a)** $1+i$ **b)** $\dfrac{-1+i\sqrt{3}}{2}$ **c)** $-\sqrt{3}-3i$ **d)** $\dfrac{1}{1+i}$ **e)** $\left(1-i\sqrt{3}\right)^n$

**14.** **(Practicing a Procedure)** Use Euler's formula to compute the real part of each of the following complex numbers in terms of trigonometric functions. Here $k$ is a positive integer.

**a)** $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^k$ **b)** $\dfrac{(2+3i)(1+i)^k}{2-i}$

**15. (Internalizing a Concept)** For each matrix $A$ and each vector $x$, decide if $x$ is an eigenvector of $A$, and if so, find the eigenvalue $\lambda$.

$$\textbf{a)}\ \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} -4 & 13 & 13 \\ 2 & -2 & -4 \\ -4 & 8 & 10 \end{pmatrix}, \begin{pmatrix} 1+5i \\ -2i \\ 4i \end{pmatrix}$$

$$\textbf{c)}\ \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ -2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2+i \\ 1 \\ -i \end{pmatrix}$$

Careful! It is difficult to recognize by inspection if two complex vectors are (complex) scalar multiples of each other—you have to divide one coordinate by the other to figure out what the eigenvalue would have to be.

**16. (Practicing a Procedure)** For each $2 \times 2$ matrix $A$, **i)** compute the characteristic polynomial, **ii)** find all (complex) eigenvalues, and **iii)** find a basis for each eigenspace using HW9#4. **iv)** Is the matrix diagonalizable (over the complex numbers)? If so, find an invertible matrix $C$ and a diagonal matrix $D$ such that $A = CDC^{-1}$.

$$\textbf{a)}\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} -3 & 5 \\ -10 & 7 \end{pmatrix}$$

**17. (Practicing a Procedure)** Diagonalize the following matrix over the complex numbers *by hand*:[1]

$$A = \begin{pmatrix} 1 & 4 & -6 \\ -6 & 7 & -22 \\ -2 & 1 & -5 \end{pmatrix}.$$

[1] This problem is included to make you do Gaussian elimination by hand with complex numbers *one time*, so that you'll be grateful to have computers do it for you in the future.

**18.** **(Practicing a Procedure)** A certain forest contains a population of rabbits and a population of foxes. If there are $r_n$ rabbits and $f_n$ foxes in year $n$, then

$$\begin{aligned} r_{n+1} &= 3r_n - f_n \\ f_{n+1} &= r_n + 2f_n: \end{aligned}$$

in other words, each rabbit produces three baby rabbits on average, but there is some loss due to predation by foxes; each fox produces two babies on average, but this is increased with ample prey.

**a)** Let $v_n = \binom{r_n}{f_n}$. Find a matrix $A$ such that $v_{n+1} = Av_n$.

**b)** Find an eigenbasis of $A$. (The eigenvectors and eigenvalues will be complex.)

[**Hint:** Part **d)** will be easier if you choose the eigenvectors with first coordinate equal to 1.]

**c)** Suppose that $r_0 = 2$ and $f_0 = 1$. Find closed formulas for $r_n$ and $f_n$. Find a formula for $r_n$ involving only real numbers. (This latter formula can involve an arctan.)

In general, any $2 \times 2$ difference equation with a complex eigenvalue will exhibit oscillation centered at zero. This phenomenon can be described explicitly, but is beyond the scope of this course.

**19.** **(Examples Problem)** Find examples of real $2 \times 2$ matrices $A$ with the following properties.

**a)** $A$ is invertible and diagonalizable over the real numbers.

**b)** $A$ is invertible but not diagonalizable over the complex numbers.

**c)** $A$ is diagonalizable over the real numbers but not invertible.

**d)** $A$ is neither invertible nor diagonalizable over the complex numbers.

This shows that *invertibility and diagonalizability have nothing to do with each other*.

**20.** **(Examples Problem)** Give an example of each of the following, or explain why no such example exists. All matrices should have real entries.

**a)** A $3 \times 3$ matrix with eigenvalues $0, 1, 2$, and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

**b)** A $4 \times 4$ matrix having eigenvalue 2 with algebraic multiplicity 2 and geometric multiplicity 3.

**c)** A $3 \times 3$ matrix with one complex (non-real) eigenvalue and two real eigenvalues.

**d)** A $2 \times 2$ matrix with two linearly independent 1-eigenvectors.

**e)** A $2 \times 2$ matrix $A$ such that $A^2$ is diagonalizable over the real numbers but $A$ is not diagonalizable, even over the complex numbers.
[**Hint:** try a nonzero matrix $A$ such that $A^2 = 0$.]

**21.** **(Foreshadowing)** Let $A$ be an $n \times n$ matrix.

**a)** Show that the product of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\det(A)$.

**b)** [**Optional**] Show that the sum of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\mathrm{Tr}(A)$.

(Both of these are identities only involving the characteristic polynomial of $A$.)

**22.** **(True-False)** Decide if each statement is true or false. If it is true, explain why; if it is false, provide a counterexample.

**a)** An $n \times n$ matrix is diagonalizable if it has $n$ eigenvalues, counted with algebraic multiplicity.

**b)** Any $2 \times 2$ real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

**c)** Any $3 \times 3$ real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

**d)** Any $4 \times 4$ real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

**e)** Any $n \times n$ matrix has a (real or complex) eigenvalue.

**f)** If the characteristic polynomial of $A$ is $-(\lambda^3 - 1) = -(\lambda^2 + \lambda + 1)(\lambda - 1)$, then the 1-eigenspace of $A$ is a line.