Math 218D-1: Homework #11

due Wednesday, November 12, at 11:59pm

1. **(Internalizing a Definition)** For each matrix, decide if it is stochastic, positive stochastic, or not stochastic.

$$\textbf{a)}\begin{pmatrix} .3 & .1 & .2 \\ .4 & .4 & .4 \\ .3 & .5 & .4 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{c)}\begin{pmatrix} .3 & .4 \\ .4 & .3 \\ .3 & .3 \end{pmatrix}$$

$$\textbf{d)}\ \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \textbf{e)}\begin{pmatrix} .3 & -.1 & .2 \\ .4 & .6 & .4 \\ .3 & .5 & .4 \end{pmatrix} \qquad \textbf{f)}\begin{pmatrix} .3 & 0 & .2 \\ .4 & 0 & .4 \\ .3 & 0 & .4 \end{pmatrix}$$

2. **(Practicing a Procedure)** For each positive stochastic matrix A and each vector v_0, **a)** find the steady state vector w of A, and **b)** compute $\lim_{k\to\infty} A^k v_0$ (this requires no additional work).

$$\textbf{a)}\ A = \begin{pmatrix} .64 & .54 \\ .36 & .46 \end{pmatrix},\ v_0 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \textbf{b)}\ A = \frac{1}{40}\begin{pmatrix} 13 & 11 & 8 \\ 5 & 19 & 8 \\ 22 & 10 & 24 \end{pmatrix},\ v_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\textbf{c)}\ A = \frac{1}{150}\begin{pmatrix} 38 & 6 & 9 & 34 \\ 54 & 78 & 57 & 42 \\ 23 & 21 & 54 & 4 \\ 35 & 45 & 30 & 70 \end{pmatrix},\ v_0 = \begin{pmatrix} 3 \\ -1 \\ -1 \\ -2 \end{pmatrix}$$

3. **(Practicing a Procedure)** Pretend that there are four car rental agencies in Durham. Suppose that a customer renting a car from agency i will return the car the next day to agency j, with the following probabilities:

Returning to agency \ Renting from agency	1	2	3	4
1	22.8%	9.2%	2.4%	0.4%
2	19.6%	44.4%	16.8%	22.8%
3	8.4%	7.6%	27.2%	11.2%
4	49.2%	38.8%	53.6%	65.6%

For instance, a customer renting from agency 3 has a 53.6% probability of returning it to agency 4.

If there are 100 cars available for rental, how many cars will be at each agency after a long time?

4. Evaluate

$$\lim_{k\to\infty}\begin{pmatrix} .3 & .1 & .2 \\ .4 & .4 & .4 \\ .3 & .5 & .4 \end{pmatrix}^k \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} =$$

without doing any computations.

5. **(Examples Problem)** In each part, find an example or explain why no such example exists.

a) A 2×2 stochastic matrix whose 1-eigenspace is a plane.

b) A 2×2 stochastic matrix with eigenvalue -1.

c) A 3×3 positive stochastic matrix with eigenvalue -1.

6. **(True-False)** Decide if each statement is true or false, and explain why.

a) A positive stochastic matrix has a 1-eigenvector whose coordinates are all *negative*.

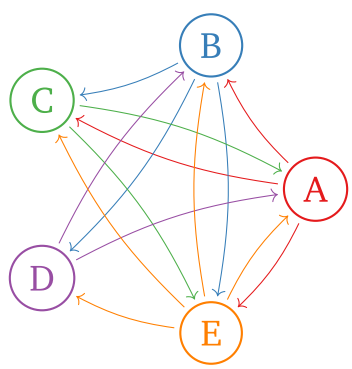
b) The 1-eigenspace of a positive stochastic matrix can be a plane.

c) If $\lambda \neq 1$ is an eigenvalue of a positive stochastic matrix, then $|\lambda| < 1$.

d) If $\lambda \neq 1$ is an eigenvalue of a positive stochastic matrix and v is a λ-eigenvector, then the coordinates of v sum to zero.

e) A positive stochastic matrix is diagonalizable.

7. **(Practicing a Procedure)** Consider the following Internet with five pages:



a) Compute the importance matrix A.

b) Compute the Google matrix G with damping factor $p = 0.15$.

c) Find the PageRank vector (with the help of a computer). Which page is the most important?

8. (Practicing a Procedure) For each symmetric matrix S, find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

$$\textbf{a)}\ \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}$$

SymPy does not (yet) have a method for orthogonally diagonalizing a symmetric matrix. You can use `S.diagonalize(normalize=True)` to produce unit eigenvectors, but eigenvectors with the same eigenvalue need not be orthogonal, so you'll still have to do Gram–Schmidt. Still, this will produce an orthogonal diagonalization when S has distinct eigenvalues.

9. (Practicing a Procedure) Consider the matrix

$$S = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

of Problem 8(d). Write S in the form $\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$ for numbers $\lambda_1, \lambda_2, \lambda_3$ and orthonormal vectors u_1, u_2, u_3.

10. (Internalizing a Concept) Find *all possible* orthogonal diagonalizations

$$\frac{1}{5}\begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} = QDQ^T.$$

How many are there?

11. (Internalizing a Concept)

a) Let S be a matrix that has a (real) orthonormal eigenbasis. Prove that S is symmetric.

[**Hint:** This is explained briefly in the lecture notes.]

b) Let S be a matrix that can be written in the form

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

for some vectors $q_1, q_2, \ldots, q_n$. Prove that S is symmetric.

[**Hint:** Do matrix algebra: symmetric means $S = S^T$.]

c) Let V be a subspace of $\mathbf{R}^n$, and let P_V be the projection matrix onto V. Use **a)** or **b)** to prove that P_V is symmetric.

[**Hint:** The eigenspaces of P_V are orthogonal—why?]

12. **(Practicing a Procedure)** Which of the following symmetric matrices are positive definite?

$$\textbf{a)}\ \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & -8 \\ -1 & -8 & 22 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & -8 \\ -1 & -8 & 16 \end{pmatrix}$$

$$\textbf{c)}\ \begin{pmatrix} 2 & -4 & 0 & 0 \\ -4 & 11 & 0 & -9 \\ 0 & 0 & 9 & 0 \\ 0 & -9 & 0 & 29 \end{pmatrix} \qquad \textbf{d)}\ \begin{pmatrix} 0 & -4 & 0 & 0 \\ -4 & 11 & 0 & -9 \\ 0 & 0 & 4 & 0 \\ 0 & -9 & 0 & 25 \end{pmatrix}$$

13. **(Practicing a Procedure)** For which values of h is the matrix

$$S = \begin{pmatrix} 1 & 3 \\ 3 & h \end{pmatrix}$$

positive-definite?

14. **(Exploration Problem)** Let S be a symmetric orthogonal 2×2 matrix.

a) Show that $S = \pm I_2$ if it has only one eigenvalue.
[**Hint:** See HW9#7.]

b) Suppose that S has two eigenvalues. Show that S is the matrix for the reflection over a line L in $\mathbf{R}^2$. (Recall that the reflection over a line L is given by $R_L = I_2 - 2P_{L^\perp}$.)
[**Hint:** Write S as $\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$, and use the projection formula to write I_2 and $P_{L^\perp}$ in this form as well. What is L?]

15. **(Internalizing a Concept)** For which matrices A is $S = A^T A$ positive-definite? If S is not positive-definite, find a vector x such that $x^T S x = 0$. In any case, do not compute S!

$$\textbf{a)}\ \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

16. **(Internalizing a Concept)**

a) If S is positive-definite and C is invertible, show that CSC^T is positive-definite.

b) If S and T are positive-definite, show that $S + T$ is positive-definite.

c) If S is positive-definite, show that S is invertible and that S^{-1} is positive-definite.

[**Hint:** For **a)** and **b)** use the positive-energy characterization of positive-definiteness; for **c)** use the positive-eigenvalue characterization.]

17. **(Exploration Problem)** Let S be a positive-definite matrix.

a) Show that the diagonal entries of S are positive.

[**Hint:** Compute $e_i^T S e_i$ and use the positive-energy criterion.]

b) Show that $S - aI_n$ is positive-definite if and only if a is smaller than the smallest eigenvalue of S.

[**Hint:** What are the eigenvalues of $S - aI_n$? Compare HW9#3(b).]

c) Show that the diagonal entries of S are all greater than or equal to the smallest eigenvalue of S.

[**Hint:** If not, apply **a)** and **b)** to $S - aI_n$ for a diagonal entry a that is smaller than all eigenvalues.]

18. **(Examples Problem)** In each part, find an example of a matrix with the stated property, or explain why no such matrix exists. All matrices must have *real entries.*

a) A symmetric matrix with eigenvalue $1 + i$.

b) A symmetric matrix that is not positive-definite but has positive determinant.

c) A symmetric matrix S satisfying

$$S \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad S \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0.$$

d) A positive-definite projection matrix.