Math 218D-1: Homework #13

due Tuesday, November 25, at 10:29pm

1. (Practicing a Procedure) For each matrix A, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

a) $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$

d) $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$ **e)** $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$

Check your answers using SymPy:

```
A = Matrix([[8,  4],
            [1, 13]])
pprint(A.singular_value_decomposition())
```

The output is a triple of matrices (U, Σ, V), where the columns of U are the u_i, the diagonal entries of Σ are the σ_i, and the columns of V are the v_i. (This is essentially the matrix form of the SVD.) Note that SymPy may produce a slightly different SVD than you: see Problem 2.

2. (Internalizing a Concept) Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 1(a). Let σ_1, σ_2 be the singular values of A. Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$. How many are there?

3. (Internalizing a Concept) Let A be a matrix with nonzero *orthogonal* columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of A in outer product form.

4. (Internalizing a Concept) Let $u_1, u_2, \ldots, u_r \in \mathbf{R}^m$ and $v_1, v_2, \ldots, v_r \in \mathbf{R}^n$ be any vectors, and let $\sigma_1, \sigma_2, \ldots, \sigma_r \in \mathbf{R}$ be any scalars. Consider the matrix

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

a) Show that Col(A) is contained in Span$\{u_1, u_2, \ldots, u_r\}$.

b) Explain why rank(A) $\leq r$.

(You're not meant to use the SVD to answer these questions.)

5. **(Internalizing a Concept)** Let S be a symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Let $\{v_1, \dots, v_n\}$ be an orthonormal eigenbasis, where v_i has eigenvalue λ_i.

a) Show that the singular values of S are $|\lambda_1|, \dots, |\lambda_r|$. In particular, $\text{rank}(S) = r$.

b) Find the singular value decomposition of S in outer product form, in terms of the λ_i and the v_i.

6. **(Synthesizing New and Old Concepts)**

a) Show that all singular values of an orthogonal matrix are equal to 1.

b) Let A be an $m \times n$ matrix, let Q_1 be an $m \times m$ orthogonal matrix, and let Q_2 be an $n \times n$ orthogonal matrix. Show that A has the *same singular values* as $Q_1 A Q_2$. [**Hint:** Use HW10#6.]

Remark: This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

7. **(Synthesizing New and Old Concepts)** Let A be a matrix of full column rank and let $A = QR$ be the QR decomposition of A.

a) Show that A and R have the same singular values $\sigma_1, \dots, \sigma_r$ and the same right singular vectors $v_1, \dots, v_r$.

b) What is the relationship between the left singular vectors of A and R?

8. Recall that the *matrix norm* of a matrix A is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$, and is denoted $\|A\|$.

a) Show that $\|Ax\|$ is maximized at the first right singular vector v_1 of A (subject to $\|x\| = 1$), and that the maximum value of $\|Ax\|$ is the first singular value σ_1.

b) Show that the maximum value of $\|Ax\|$, subject to $\|x\| = 1$ and $x \perp v_1$, is attained at the second right singular vector v_2, and that the second-largest value of $\|Ax\|$ is the second singular value σ_2.

c) Suppose now that A is square and λ is an eigenvalue of A. Use **a)** to show that $|\lambda| \leq \sigma_1$. (You may assume λ is real, although it is also true for complex eigenvalues.)

[**Hint:** Compute $\|Av\|$, where v is a unit λ-eigenvector.]

This shows that *the largest singular value is at least as big as the largest eigenvalue.*

9. (Exploration Problem)

a) Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 \end{pmatrix}.$$

c) Note that A is very close to A' numerically. Were the eigenvalues of A close to the eigenvalues of A'? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are numerically stable*. This is another advantage of the SVD.

10. (Foreshadowing) Let A be a matrix with singular values $\sigma_1, \sigma_2, \ldots, \sigma_r$. Show that the following four quantities are equal:

$$\begin{aligned} \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2 &= \mathrm{Tr}(A^T A) = \mathrm{Tr}(AA^T) \\ &= (\text{the sum of the squares of the entries of } A). \end{aligned}$$

[**Hint:** Use HW10#21(b) for the first two equalities. For the last one, write out $\mathrm{Tr}(A^T A)$.]

11. (Examples Problem) In each case, find an example or explain why none exists.

a) A nonzero 3×2 matrix A such that $A^T A$ and AA^T have the same eigenvalues.

b) A nonzero 3×3 matrix A having a right singular vector contained in $\mathrm{Nul}(A)$.

c) A nonzero 3×3 matrix whose singular values are its eigenvalues.

d) A nonzero 3×3 matrix with singular value 0.

e) A nonzero 3×3 matrix that is a linear combination of three rank-1 matrices.

f) A nonzero 2×2 matrix with infinitely many different singular value decompositions.

12. (Practicing a Procedure) For each matrix A of Problem 1:

$$\textbf{a)}\ \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

find the singular value decomposition in the matrix form

$$A = U\Sigma V^T.$$

13. (Internalizing a Concept) The matrix

$$A = \begin{pmatrix} 3 & -1 & 2 \\ -1 & -3 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

has singular value decomposition $A = U\Sigma V^T$ where

$$U = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \qquad V = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5} \\ 2/3 & 0 & 5/3\sqrt{5} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 3\sqrt{3} & 0 & 0 \\ 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Find the rank of A.

b) Write an orthonormal basis of each of the four subspaces of A.

c) Orthogonally diagonalize the matrices A^TA and AA^T.

d) Orthogonally diagonalize the projection matrices P_W for $W = \mathrm{Col}(A)$ and $W = \mathrm{Row}(A)$.[1]

(Orthogonally diagonalizing S means finding a decomposition $S = QDQ^T$ for an orthogonal matrix Q and a diagonal matrix D.) You can extract all of the information you need for this problem from $A = U\Sigma V^T$; do not do any additional computation!

[1] I'm not using P_V because the letter V is already taken.

14. (Internalizing a Concept)

a) Let A be an invertible $n \times n$ matrix. Show that the product of the singular values of A equals the absolute value of the product of the (real and complex) eigenvalues of A (counted with algebraic multiplicity).

[**Hint:** Both equal $|\det(A)|$. What is $\det(A^T A)$?]

b) Find an example of a 2×2 matrix A with distinct positive eigenvalues that are not equal to any of the singular values of A.

[**Hint:** One of the matrices in Problem 1 works.]

15. (Internalizing a Concept) Let A be a square, invertible matrix with singular values $\sigma_1, \ldots, \sigma_n$.

a) Show that A^{-1} has the same singular vectors as A^T, with singular values $\sigma_n^{-1} \geq \cdots \geq \sigma_1^{-1}$. [**Hint:** invert $A = U\Sigma V^T$.]

b) Let λ be an eigenvalue of A. Use Problem 8(b) and **a)** to show that $\sigma_n \leq |\lambda|$.

It follows that the absolute values of all eigenvalues of A are contained in the interval $[\sigma_n, \sigma_1]$. Compare Problem 14.

16. (Exploration Problem) Let A be an invertible $n \times n$ matrix. The *condition number* of A is defined to be

$$\kappa(A) = \|A\|\,\|A^{-1}\|,$$

where $\|\cdot\|$ denotes the matrix norm (see Problem 8).

a) If x is any nonzero vector, show that $\|Ax\|/\|x\| \leq \|A\|$.

[**Hint:** Pull the $1/\|x\|$ into $\|Ax\|$.]

b) Use Problem 8 and Problem 15 to show that $\kappa(A) = \sigma_1/\sigma_n$, where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of A.

c) Use **b)** to show that $\kappa(A) \geq 1$.

Suppose that you have measured a vector b_0 to within an error of $\varepsilon > 0$—in other words, you have found a vector $b = b_0 + e$ for $\|e\| < \varepsilon$. If you want to solve $Ax_0 = b_0$, then you end up solving $Ax = b$; the actual solution is $x_0 = A^{-1}b_0$, but the solution that you obtain is

$$x = A^{-1}b = A^{-1}(b_0 + e) = A^{-1}b_0 + A^{-1}e = x_0 + A^{-1}e.$$

In other words, the error in the solution you obtained is $A^{-1}e$.

d) Use **a)** to show that

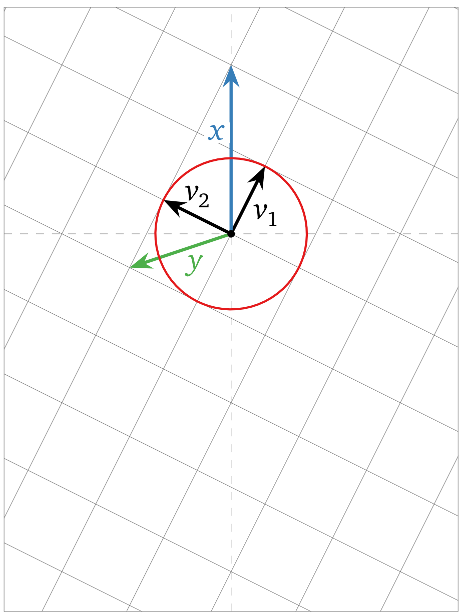
$$\frac{\|A^{-1}e\|}{\|x_0\|} \leq \kappa(A)\,\frac{\|e\|}{\|b_0\|}.$$

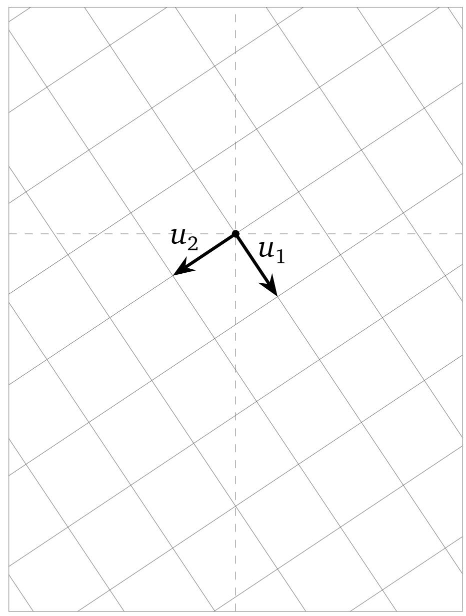
You should interpret $\|A^{-1}e\|/\|x_0\|$ as the amount of error in the solution of $Ax = b$ relative to the length of the solution x_0, and $\|e\|/\|b_0\|$ as the amount of error in the measurement of b_0 relative to the length of b_0—both measure the number of *digits of precision*. The inequality above says that if A is *ill-conditioned*, which means that $\kappa(A)$ is large, then *small errors in b lead to large errors in x*. Said differently, if you want to solve $Ax = b$ to a certain number of decimal places in x, then you have to know b to a much higher level of precision. This is why the condition number is a very important numerical invariant of A.

17. **(Picture Problem)** A certain 2×2 matrix A has singular values $\sigma_1 = 2$ and $\sigma_2 = 1.5$. The right-singular vectors v_1, v_2 and the left-singular vectors u_1, u_2 are shown in the pictures below.

a) Draw Ax and Ay in the picture on the right.

b) Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit circle by A) in the picture on the right.

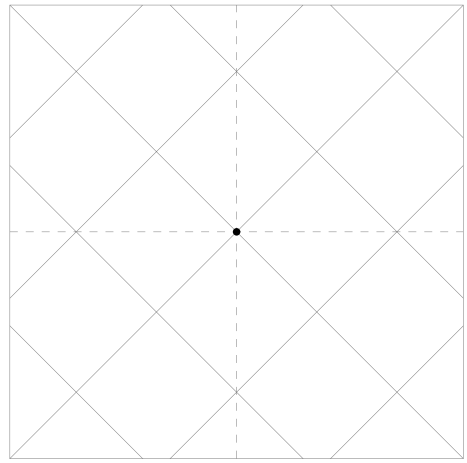




18. **(Picture Problem)** Consider the following 3×2 matrix A and its SVD:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^T.$$

Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit sphere by A) in the picture on the right.



19. **(Picture Problem)** A certain matrix A has 2 rows and 10 columns. Its SVD has the form

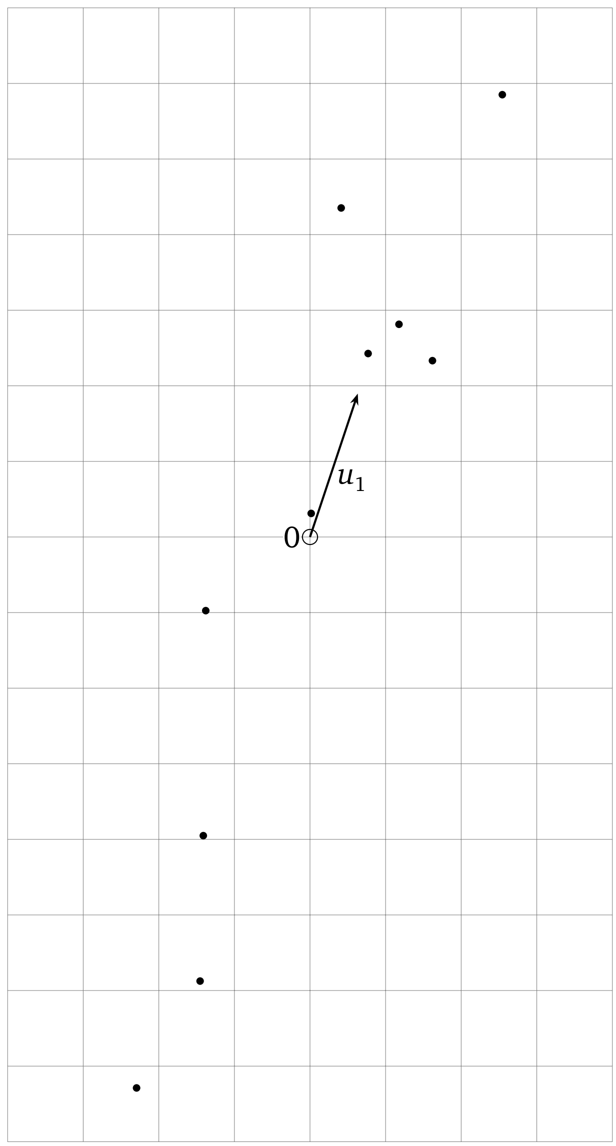
$$A = 7u_1 v_1^T + 0.9u_2 v_2^T,$$

where u_1, u_2 and v_1, v_2 are the singular vectors. The columns of A and the first left singular vector u_1 are drawn below. *Draw and label*:

a) the span of u_1;

b) the columns of $7u_1 v_1^T$ (drawn as dots);

c) the columns of $0.9u_2 v_2^T$ (drawn as arrows).



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