Math 218D-1: Homework #2

due Wednesday, September 10, at 11:59pm

Once you are comfortable doing the Gauss–Jordan elimination algorithm(s) by hand, *please* start using SymPy on the Sage cell on the course webpage to do the computations! Just write "used SymPy" so that you don't confuse the graders. This class is about formulating linear algebra problems that a computer can solve, not mastering computations that a computer can do better than you. (You will still need to do computations by hand on exams.)

Sage cell tips:

```
  # Specify a matrix
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]])
  # Shorthand for specifying a column vector
b = Matrix([1, 2, 3])
  # Solve Ax=b (only works when there's a unique solution)
pprint(A.solve(b))
  # Or, augment [A|b] and find the rref:
pprint(A.row_join(b).rref(pivots=False))
```

1. Find values of a and b such that the following system has **a)** zero, **b)** exactly one, and **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - y &= b \end{aligned}$$

[Find the relevant criterion involving pivots in the notes.]

2. Give examples of matrices A in *reduced row echelon form* for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?

3. **(Practicing a Procedure)** For each matrix A and vector b, decide if the system $Ax = b$ is consistent. If so, find the parametric vector form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Also answer the following questions (for the systems that have solutions): Which variables are free? How many solutions does the system have? What is the dimension of the solution set?

a) $$A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

c) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 48 \end{pmatrix}$$

d) $$A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

e) $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

You can check your work again using SymPy. When $Ax = b$ has infinitely many solutions, `A.solve(b)` will throw an error; instead, try this:

```
A = Matrix([[2, 1, 1, 4],
            [4, 2, 1, 7]])
b = Matrix([1, 1])
  # Find the parametric form (free variables are labelled
  # tau0, tau1, ...)
pprint(A.gauss_jordan_solve(b))
  # Or, form the augmented matrix (A|b) and find its rref,
  # then do the rest by hand:
pprint(A.row_join(b).rref(pivots=False))
```

4. Is $\begin{pmatrix}3\\3\\3\end{pmatrix}$ a linear combination of $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}4\\5\\6\end{pmatrix}$, $\begin{pmatrix}7\\8\\9\end{pmatrix}$? If so, what are the weights?

[Translate the problem into a linear algebra problem that you can solve.]

5. **(Foreshadowing)** Find the parametric vector form of the solution sets of the following systems of equations:

$$\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ 4x_1 + 2x_2 + x_3 = 0 \end{cases} \qquad \begin{cases} 2x_1 + x_2 + x_3 = 1 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases}$$

How are the solution sets related to each other geometrically?

6. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix}1+t\\2-t\\t\end{pmatrix} \qquad t \in \mathbf{R}.$$

7. Suppose that A is a 3×3 matrix and b is a vector such that the solution set of $Ax = b$ is a line in $\mathbf{R}^3$. How many pivots does A have?

8. **(Examples Problem)** In each part, find an example of a matrix with the stated property, or explain why no such matrix exists.

a) A 3×3 matrix with one free variable.

b) An invertible 3×3 matrix with one free variable.

c) A 2×3 matrix with 3 pivots.

d) A 2×3 matrix with no free variables.

e) A 3×2 matrix A such that $Ax = (1, 1, 1)$ has infinitely many solutions.

f) An invertible 2×2 matrix A such that A^3 is not invertible.

9. **(Practicing a Procedure)** Use the formula for the 2×2 inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

$$\textbf{a)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

10. **(Practicing a Procedure)** Compute the inverses of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why. You're welcome to use Rabinoff's Reliable Row Reducer, but *write out all row operations you perform.*

$$\textbf{a)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 4 \\ -3 & 1 & 4 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 6 & -4 & -7 & -1 \\ 7 & 0 & 1 & 3 \\ -1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

Check your answers in SymPy, as in:

```
A = Matrix([[1, 1, 0],
            [1, 2, 1],
            [0, 1, 2]])
pprint(A.inv())
```

11. Consider the linear system

$$\begin{aligned} x_1 + x_2 \phantom{{}+2x_3} &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the Problem 10(a) to solve for x_1, x_2, x_3 in terms of b_1, b_2, b_3. Do *not* use Gauss–Jordan elimination!

[Find the relevant big red box in the notes.]

12. Suppose that

$$A\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

What is A^{-1}?

[**Hint:** multiply both sides by A^{-1}. This requires no computations.]

13. Suppose that A, B, and C are invertible 3×3 matrices. Simplify the following expressions (write them without parentheses or unnecessary identity matrices):

a) $(ABC)^{-1}$ **b)** $C(A - 2I_3)C^{-1}$ **c)** $A^T(A^{-1})^T$ **d)** $A^3(A^{-1})^2$

14. **(Internalizing a Definition)** Write the elementary matrices that perform the following row operations on a 3×4 matrix:

a) $R_2 \mathrel{+}= 2R_1$ **b)** $R_1 \mathrel{-}= \frac{1}{2}R_3$ **c)** $R_3 \mathrel{\times}= 2$

d) $R_3 \mathrel{\div}= 2$ **e)** $R_1 \longleftrightarrow R_3$ **f)** $R_1 \longleftrightarrow R_2$

15. **(Internalizing a Definition)** Write the row operations that the following elementary matrices perform:

a) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ **e)** $\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

16. For each elementary matrix in Problem 15, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row op}} R_2 \mathrel{+}= R_1 \xrightarrow{\text{undo}} R_2 \mathrel{-}= R_1 \xrightarrow{\text{matrix}} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

17. **(Internalizing a Definition)** Decide if each matrix is upper-triangular, upper-unitriangular, lower-triangular, lower-unitriangular, diagonal, not triangular, or some combination of these. (For instance, any upper-unitriangular matrix is also upper-triangular.)

a) $\begin{pmatrix} 1 & 3 & 2 & 7 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 2 & 4 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 0 & 4 \end{pmatrix}$ **f)** $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **g)** $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

18. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

a) Explain how to reduce A to a matrix U in REF using three row replacements.

b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i:

$$U = \_\_\_\_\_\_\_\_ A.$$

c) Fill in the blank with a product involving the E_i^{-1}:

$$A = \_\_\_\_\_\_\_\_ U$$

d) Evaluate that product to produce a lower-unitriangular matrix L such that $A = LU$.

When multiplying elementary matrices, just perform row operations!

e) Explain how to reduce U to the 3×3 identity matrix using three more row operations E_4, E_5, E_6.

f) Fill in the blank with a product involving the E_i:

$$A^{-1} = \_\_\_\_\_\_\_\_\_\_\_\_.$$

19. **(Practicing a Procedure)** Compute the $A = LU$ decomposition of the following matrices using the 2-column method. Check your answers by multiplying LU.

a) $\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$

Check your work again in SymPy. For instance, in **a)** you would do something like:

```
A = Matrix([[ 2,   3,   4],
            [-2,   0,  -2],
            [-6, -15, -17]])
L, U, _ = A.LUdecomposition()
pprint(L)
pprint(U)
```

20. **(Practicing a Procedure)** Solve the following matrix equations by substitution, using the provided LU decomposition. Check your answers by evaluating Ax. *Show your work.*

a)

$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

b)

$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

c)

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

You can also check your answers in SymPy, as in:

```
L = Matrix([[ 1,  0, 0],
            [-2,  1, 0],
            [-1, -2, 1]])
U = Matrix([[3,  2, 7],
            [0, -1, 4],
            [0,  0, 2]])
b = Matrix([14, -26, -16])
y = L.lower_triangular_solve(b)
pprint(y)
x = U.upper_triangular_solve(y)
pprint(x)
```

21. **(Practicing a Procedure)** Compute a $PA = LU$ decomposition for each of the following matrices, using the 3-column method and performing *maximal partial pivoting*. Check your answers by multiplying PA and LU.

a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$

22. **(Exploration Problem)** Recall that a *permutation matrix* is a product of elementary matrices for row swaps.

a) Let E be the $n \times n$ elementary matrix for the row swap $R_i \longleftrightarrow R_j$. Which entries of E are different from I_n?

b) If E is the $n \times n$ elementary matrix for a row swap, explain why $E = E^{-1}$ and why $E = E^T$ (E is *symmetric*).

c) If P is any permutation matrix, show that $P^{-1} = P^T$.
[**Hint:** write $P = E_1 E_2 \cdots E_r$ for elementary row swaps E_i and take transposes.]

d) Is $P = P^T$ for a general permutation matrix? Explain why, or give a counterexample.

23. **(Practicing a Procedure)** Solve the following matrix equations by substitution, using the provided $PA = LU$ decomposition. Check your answers by evaluating Ax.

a)

$$\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} x = \begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

You can also check your answers in SymPy, as in:

```
L = Matrix([[ 1,  0, 0],
            [ 4,  1, 0],
            [-4, -1, 1]])
U = Matrix([[-5, 4,  0],
            [ 0, 3,  0],
            [ 0, 0, -5]])
  # This is the permutation matrix whose rows are:
  #   1. the last row of the identity (index 2),
  #   2. the middle row of the identity (index 1),
  #   3. the first row of the identity (index 0)
P = eye(3).perm([2, 1, 0])
b = Matrix([54, -59, -14])
y = L.lower_triangular_solve(P*b)
pprint(y)
x = U.upper_triangular_solve(y)
pprint(x)
```

24. **(Driving a Point Home)** Suppose that A is a one million by one million matrix, and that you have to solve $Ax = b$ for one million values of b. About how many times faster can your computer solve this problem using LU decompositions than by doing elimination one million times?