Math 218D-1: Homework #3

due Wednesday, September 17, at 11:59pm

Starting this week, you will find that the material becomes more and more *conceptual*. Naturally, the homework problems will be more conceptual as well. If you're at a loss as to how to approach a problem, here is some basic advice.

(1) Did you understand the *definitions* of the words in the problem? If the problem is "list five nonzero vectors contained in this span," and you only have a vague idea of what a span is, then you'll have trouble answering that question.

Find the definition in **the notes** and go from there. They're all written down explicitly.

(2) Can you identify *which concepts* the problem is meant to help you understand? For instance, Homework #3 focuses on the concepts in week 3. Can you find something in **the notes** that reminds you of what the problem is asking?

(3) Are you familiar with the *basic properties* of the concepts in the problem? These are usually found shortly after the definition in **the notes.** For instance, "every span contains the origin."

(4) What are the *relationships* between the *concepts in the problem* and other concepts we've seen? Important results are usually labeled "Thm" and/or are contained in a big red box in **the notes.** For instance, "$Ax = b$ is consistent $\iff$ b is in the span of the columns of A" is an extremely important property of spans, and is contained in an extremely thick red box.

(5) Don't short-cut the process. If you think you've learned something because ChatGPT's answer makes sense to you, then you're fooling yourself. (Also, you're *not allowed* to use AI to help you on the homework—see the syllabus.) Could you answer a different question about the same concept on an exam?

(6) Finally, if you're still stuck after you've tried everything you know how to do, *come to office hours* and ask questions!

Notice that most of these tips involve *looking through the notes.* I don't expect you to be able to come to class, pay attention, and absorb enough of the material to immediately solve most of the homework problems. Engaging with the lectures is only the *first step* to mastering the material.

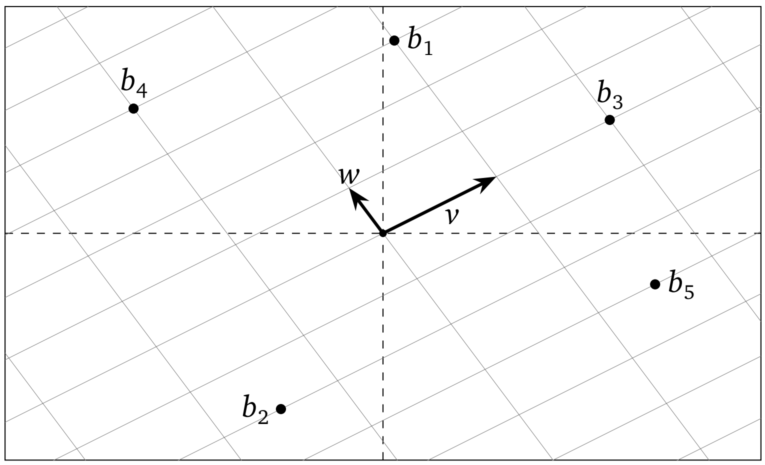
From this point on, it is not necessary to write out row operations when performing the Gauss–Jordan elimination algorithm. Once you are comfortable doing elimination by hand, *please* start using SymPy on the Sage cell on the course webpage! (And remember to write "used SymPy" when you do.)

1. (Picture Problem) Consider the vectors

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Draw the 16 linear combinations $cv + dw$ $(c, d = -1, 0, 1, 2)$ as *points* in the xy-plane. (They should all lie on the vertices of a grid.)

2. (Picture Problem) Certain vectors v, w in $\mathbf{R}^2$ are drawn below. Express each of b_1, b_2, b_3, b_4, b_5 as a linear combination of v, w. *Do not try to guess the coordinates of v and w!* This is a question about the geometry of linear combinations.



3. (Picture Problem) Consider the vectors

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Draw a picture of all of the linear combinations $au + bv$ for real numbers a, b satisfying $0 \le a \le 1$ and $0 \le b \le 1$. (This will be a shaded region in the xy-plane.)

4. Consider the span

$$V = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}.$$

a) List five nonzero vectors contained in V.

b) Is $(0, 3, 6)$ contained in V?

If so, express $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

c) Find a vector not contained in V.

5. Give geometric descriptions of the following spans (line, plane, . . .).

a) $\text{Span}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **c)** $\text{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\-6\end{pmatrix}\right\}$

d) $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ **e)** $\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$

[**Hint:** for **d)** and **e)**, how can you decide if the third vector is already in the plane spanned by the first two?]

6. (Picture Problem) Draw a picture of all vectors $b \in \mathbf{R}^2$ for which the equation

$$\begin{pmatrix}1 & 2\\-2 & -4\end{pmatrix}x = b$$

is consistent. [**Hint:** the answer is a span!]

7. (Practicing a Procedure) For each matrix A and vector b, and express the solution set in the form

$$p + \text{Span}\{???\}$$

for some vector p. For instance,

$$\begin{pmatrix}1 & -1\\2 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix}1\\0\end{pmatrix} + \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

Now express the solution set of $Ax = 0$ as a span, as in:

$$\begin{pmatrix}1 & -1\\2 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \rightsquigarrow \quad \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

Does there exist a nontrivial solution of $Ax = 0$?

[**Hint:** You found the parametric vector form in HW2#3, so this doesn't require any additional computation.]

a) $$A = \begin{pmatrix}2 & 1 & 1 & 4\\4 & 2 & 1 & 7\end{pmatrix} \qquad b = \begin{pmatrix}1\\1\end{pmatrix}$$

b) $$A = \begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix} \qquad b = \begin{pmatrix}3\\2\\49\end{pmatrix}$$

c) $$A = \begin{pmatrix}1 & 2 & 3 & -1 & 1\\-2 & -4 & -5 & 4 & 1\\1 & 2 & 2 & -3 & -1\\-3 & -6 & -7 & 7 & 6\end{pmatrix} \qquad b = \begin{pmatrix}2\\4\\-6\\10\end{pmatrix}$$

d) $$A = \begin{pmatrix}1 & 1 & 0\\1 & 2 & 1\\0 & 1 & 2\end{pmatrix} \qquad b = \begin{pmatrix}2\\5\\4\end{pmatrix}$$

8. (Internalizing a Concept) Let A be a 3×4 matrix whose columns span the plane $b_1 + b_2 + b_3 = 0$.

a) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *consistent*.

b) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *inconsistent*.

[**Hint:** this problem requires no computations at all. Find the relevant big red box in the notes.]

9. (Internalizing a Concept) For a certain 3×2 matrix A, the solution set of $Ax = 0$ is equal to $\mathrm{Span}\left\{\binom{1}{1}\right\}$. Which of the following sets is *necessarily* equal to the solution set of $Ax = b$ for *some* vector $b \in \mathbf{R}^3$? Why or why not?

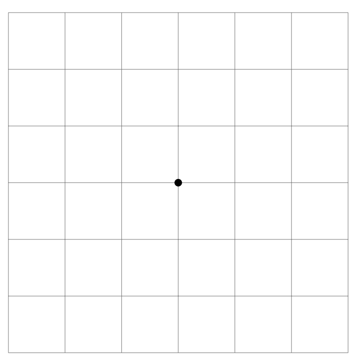
$$\textbf{a)}\ \mathrm{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} \qquad \textbf{b)}\ \{\}\ \text{(no solutions)} \qquad \textbf{c)}\ \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$$

$$\textbf{d)}\ \begin{pmatrix}0\\1\end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \qquad \textbf{e)}\ \begin{pmatrix}11\\12\end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix}2\\2\end{pmatrix}\right\}$$

[Which concept(s) are you meant to be internalizing here?]

10. (Picture Problem) Draw *and label* the solution sets of the three matrix equations $Ax = b$, $Ax = 0$, and $Ax = -b$, where

$$A = \begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix} \qquad b = \begin{pmatrix}-1\\1\end{pmatrix}.$$



11. (Examples Problem) In each part, find an example of a matrix with the stated property, or explain why no such matrix exists.

a) A 2×3 matrix A such that $Ax = \binom{1}{2}$ is consistent but $Ax = \binom{2}{1}$ is inconsistent.

b) A 2×2 matrix A such that $Ax = \binom{2}{2}$ has infinitely many solutions but $Ax = \binom{2}{-2}$ has exactly one solution.

c) A 3×2 matrix A such that the span of the columns of A does not contain the zero vector.

d) A 3×3 matrix A such that $Ax = (0, 0, 0)$ is inconsistent.

e) A 3×2 matrix A such that the solution set of $Ax = (1, 2, 3)$ is a line.

12. (Internalizing a Definition) Give examples of subsets V of $\mathbf{R}^2$ such that:

a) V is closed under addition and contains 0, but is not closed under scalar multiplication.

b) V is is closed under scalar multiplication and contains 0, but is not closed under addition.

c) V is closed under addition and scalar multiplication, but does not contain 0.

In **a)** and **b)**, find a *different* example from the one in the notes.

It follows that none of these conditions is redundant.

13. (Internalizing a Definition) Without doing *any computations at all:*

a) Find a matrix A such that

$$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\3\\2\\4\end{pmatrix}, \begin{pmatrix}2\\-1\\3\\2\end{pmatrix}\right\}.$$

b) Find vectors v_1, v_2, v_3 such that

$$\mathrm{Col}\begin{pmatrix}1 & 7 & 4\\ 2 & 2 & 3\\ 4 & 2 & 1\\ -1 & 7 & 8\end{pmatrix} = \mathrm{Span}\{v_1, v_2, v_3\}.$$

14. (Internalizing a Concept) A certain 3×3 matrix A has the property that

$$\mathrm{Col}(A) = \mathrm{Nul}\begin{pmatrix}1 & 3 & 2\\ 2 & 0 & 4\end{pmatrix}.$$

Find a *nonzero* vector $b \in \mathbf{R}^3$ such that $Ax = b$ is *consistent*.

15. **(Driving a Point Home)** Which of the following subspaces are equal to the plane

$$V = \{(x, y, z) \in \mathbf{R}^3 \; : \; x = z\}?$$

Explain.

a) $\mathrm{Nul}\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$ **b)** $\mathrm{Col}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ **c)** $\mathrm{Col}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$

d) $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right\}$ **e)** $\{(x, y, x) \; : \; x, y \in \mathbf{R}\}$

f) $\mathrm{Nul}\begin{pmatrix} 2 & 0 & -2 \\ 3 & 0 & -3 \end{pmatrix}$

16. **(Practicing a Procedure)** Express the null space of each matrix as a span.

a) $\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 6 & 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -3 \\ 0 & 2 & -2 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 1 \end{pmatrix}$

Check your answers using SymPy:

```
A = Matrix([[1, 3, 2, 4],
            [2, 6, 0, 1]])
pprint(A.nullspace())
```

17. Which of the following subsets of $\mathbf{R}^3$ are subspaces? If it is not a subspace, find a counterexample to one of the subspace properties. If it is, express it as the column space or null space of some matrix.

[**Hint:** does the subset naturally have an implicit or parametric description?]

a) The plane $\{(x, y, -x) \colon x, y \in \mathbf{R}\}$.

b) The plane $\{(x, y, 1) \colon x, y \in \mathbf{R}\}$.

c) The set consisting of all vectors (x, y, z) such that $xy = 0$.

d) The set consisting of all vectors (x, y, z) such that $x \leq y$.

e) The span of $(1, 2, 3)$ and $(2, 1, -3)$.

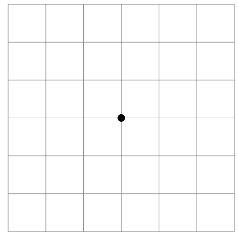
f) The solution set of the system of equations $\begin{cases} x + y = -z \\ x - 2y = z. \end{cases}$

g) $\left\{\begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} x \colon x \in \mathbf{R}^2\right\}$ **h)** $\left\{x \in \mathbf{R}^3 \colon \begin{pmatrix} 1 & 3 & 2 \\ 4 & 2 & 6 \\ 5 & 5 & 2 \end{pmatrix} x = 10x\right\}$

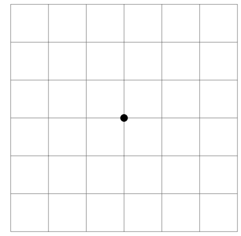
18. **(Picture Problem)** Draw pictures of the null space and the column space of the following matrices. Be precise! [**Hint:** you know how to draw a span...]

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

Nul(A)

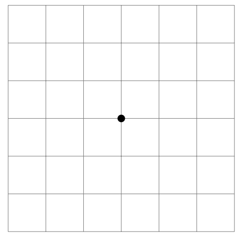


Col(A)

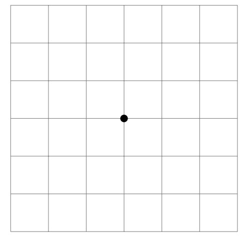


b) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

Nul(A)

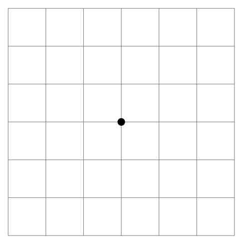


Col(A)

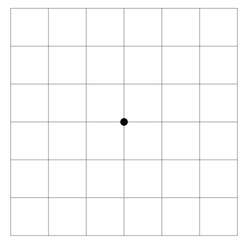


c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

Nul(A)



Col(A)



19. **(Stretch Problem)** Let A and B be two matrices that can be multiplied together.

Claim: $AB = 0$ (the zero matrix) if and only if $\mathrm{Col}(B)$ is contained in $\mathrm{Nul}(A)$. (In other words, every vector in $\mathrm{Col}(B)$ is also in $\mathrm{Nul}(A)$.)

Explain why the claim is true. (If $AB = 0$, what happens when you multiply AB by the unit coordinate vector e_i? You will have to use several concepts here.)

Then find a nonzero 2×2 matrix A such $A^2 = 0$ (which means $\mathrm{Col}(A)$ is contained in $\mathrm{Nul}(A)$, as you just showed).

20. **(Exploration Problem)** Let A and B be two matrices that can be multiplied together.

Claim: If $Bx = 0$, then $ABx = 0$.

Explain why the claim is true, and use it to show that

$$\mathrm{Nul}(B) \quad \text{is contained in} \quad \mathrm{Nul}(AB).$$

(In other words, every vector in $\mathrm{Nul}(B)$ is also in $\mathrm{Nul}(AB)$.) Take both A and B to be the matrix you found in Problem 19 to give an example where $\mathrm{Nul}(B) \neq \mathrm{Nul}(AB)$.

21. **(Exploration Problem)** Let A and B be two matrices that can be multiplied together.

Claim: If $ABx = b$ is consistent, then $Ay = b$ is consistent.

Explain why the claim is true, and use it to show that

$$\text{Col}(AB) \quad \text{is contained in} \quad \text{Col}(A).$$

(In other words, every vector in Col(AB) is also in Col(A).) Take both A and B to be the matrix you found in Problem 19 to give an example where $\text{Col}(AB) \neq \text{Col}(A)$.

22. **(True-False)** Decide if each statement is true or false. If it is true, explain why; if it is false, provide a counterexample.

a) If U is an echelon form of A, then $\text{Nul}(U) = \text{Nul}(A)$.

b) If U is an echelon form of A, then $\text{Col}(U) = \text{Col}(A)$.