Math 218D-1: Homework #4

due Wednesday, September 24, at 11:59pm

> The material in the class will become progressively more conceptual, with a larger emphasis on geometric understanding. I expect it will take you around 10 hours per week to solve the homework problems. Please don't waste your time doing elimination over and over again by hand though! Use the SymPy on the Sage cell instead.
>
> Remember, these problems are all intended to help you understand the lectures, so you should be reading through *the notes* when you get stuck. If you're still stuck after a while, come to office hours! There are links to other resources on the course website.

1. **(Practicing a Procedure)** Which sets of vectors are linearly independent, and which are linearly dependent? If the vectors are linearly dependent, find a linear relation among them.

a) $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ **b)** $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$ **c)** $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$

d) $\left\{\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix}, \begin{pmatrix}2\\-4\\2\\-6\end{pmatrix}, \begin{pmatrix}3\\-5\\2\\-7\end{pmatrix}, \begin{pmatrix}-1\\4\\-3\\7\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\6\end{pmatrix}\right\}$ **e)** $\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$

How would you solve this problem using SymPy? (Feel free to do so!) Which sets do you know are linearly dependent without doing any work?

2. Consider the vectors

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$$

of Problem 1(a).

a) Express $(5, 7, 9)$ as a linear combination of the first two vectors. How many ways are there to do this?

b) Find two different ways to express $(5, 7, 9)$ as a linear combination of all three vectors.

In particular, taking linear combinations of all three vectors does not give a *unique* parameterization of their span.

3. Certain vectors v_1, v_2, v_3, v_4 span a 3-dimensional subspace of $\mathbf{R}^5$. They satisfy the linear relation

$$2v_1 + 0v_2 - v_3 + v_4 = 0.$$

a) Describe *all* linear relations among v_1, v_2, v_3, v_4.

[**Hint:** what is the rank of the matrix with columns v_1, v_2, v_3, v_4? What is its null space? What does this have to do with the problem?]

b) Which vector(s) is/are *not* in the span of the others? How do you know for sure?

4. **(Internalizing a Concept)** Let $\{w_1, w_2, w_3\}$ be a basis for a subspace V, and set

$$v_1 = w_2 + w_3 \qquad v_2 = w_1 + w_3 \qquad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for V.

[**Hint:** You only have to check that they're linearly independent (why?). Try to solve the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ by substituting in the w_i's.]

5. **(Internalizing a Concept)** Which of the following form a basis for the plane

$$V = \left\{(x, y, z) \in \mathbf{R}^3 \ : \ 3x + 2y + z = 0\right\}?$$

Why or why not? (This is not a computational problem.)

a) $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}2\\0\\-6\end{pmatrix}\right\}$ **b)** $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\-1\\2\end{pmatrix}\right\}$ **c)** $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\}$

d) $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\2\end{pmatrix}\right\}$ **e)** $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ **f)** $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

g) $\begin{pmatrix}-1 & 0\\0 & -1\\3 & 2\end{pmatrix}$ **h)** Span $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\}$ **i)** $\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}\right\}$

6. **(Practicing a Procedure)** Find a basis for each of the following subspaces. What is the dimension of the subspace?

a) Col$\begin{pmatrix}1 & 3 & 2 & 4\\2 & 6 & 0 & 1\end{pmatrix}$ **b)** Span $\left\{\begin{pmatrix}-1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\-3\\-2\end{pmatrix}\right\}$ **c)** Nul$\begin{pmatrix}1 & 3\\2 & 4\\0 & 1\end{pmatrix}$

d) Nul$\begin{pmatrix}1 & 3 & 2 & 4\\2 & 6 & 0 & 1\end{pmatrix}$ **e)** $\left\{x \in \mathbf{R}^3 : \begin{pmatrix}0 & 6 & 8\\\frac{1}{2} & 0 & 0\\0 & \frac{1}{2} & 0\end{pmatrix} x = 2x\right\}$

f) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

g) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

7. Let A be a 3×3 matrix of rank 2. Use HW3#19 to explain why A^2 is not the zero matrix.

8. **(Driving a Point Home)** Find a basis for

$$\mathrm{Col}\begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 1 \end{pmatrix}$$

that does not involve a scalar multiple of any column of the matrix.

9. **(Examples Problem)** In each part, find an example or explain why no such example exists.

- **a)** Vectors $v_1, v_2, v_3, v_4 \in \mathbf{R}^5$ that are linearly independent, such that their span has dimension 3.
- **b)** Vectors $v_1, v_2, v_3 \in \mathbf{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly dependent but v_3 is not a linear combination of v_1 and v_2.
- **c)** Vectors $v_1, v_2, v_3, v_4 \in \mathbf{R}^5$ such that the set $\{v_1, v_2, v_3\}$ is linearly dependent but $\{v_1, v_2, v_3, v_4\}$ is linearly independent.
- **d)** A 4×5 matrix with linearly independent columns.
- **e)** A 3×3 matrix in row echelon form with rank 2.
- **f)** Two nonzero vectors contained in the plane $V = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ that do *not* form a basis.
- **g)** A 3-dimensional subspace of $\mathbf{R}^3$ that is not equal to all of $\mathbf{R}^3$.

10. **(Practicing a Procedure)** Find a basis for each of the four fundamental subspaces of each matrix, and compute their dimensions. Verify that:
(1) $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A)$ is the number of columns of A.
(2) $\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A^T)$ is the number of rows of A.
(3) $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$.
[**Hint:** Augment with the identity matrix so you only have to do Gauss–Jordan elimination once.]

a) $\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

As always, you're more than welcome to do elimination in SymPy if you're already comfortable doing it by hand. Here's one way you could augment by the identity matrix and eliminate:

```
  # Augment with the 2x2 identity matrix eye(2)
A = Matrix([[2, 1, 1, 4],
            [4, 2, 1, 7]]).row_join(eye(2))
pprint(A.rref())
```

11. **(Synthesizing New and Old Concepts)** Suppose that A is an invertible 4×4 matrix. Find bases for its four fundamental subspaces.

[**Hint:** No calculations are necessary.]

12. **(Examples Problem)** Find an example of a matrix with the required properties, or explain why no such matrix exists.

a) The column space contains $(1,2,3)$ and $(4,5,6)$, and the row space contains $(1,2)$ and $(2,3)$.

b) The column space has basis $\{(1,2,3)\}$, and the null space has basis $\{(3,2,1)\}$.

c) The dimension of the null space is one greater than the dimension of the left null space.

d) A 3×5 matrix whose row space equals its null space.

e) A matrix A such that

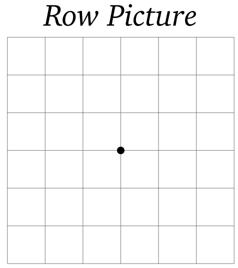
$$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}\right\} \quad \text{and} \quad \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}.$$

f) A 5×4 matrix of rank 3 whose left null space has dimension 3.

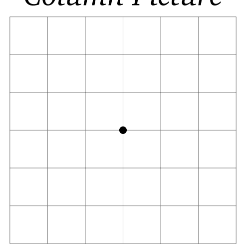
13. **(Picture Problem)** Draw the four fundamental subspaces of the following matrices, in grids like below. Be precise!

a) $\begin{pmatrix}1 & 3\\2 & 6\end{pmatrix}$ **b)** $\begin{pmatrix}1 & 0\\2 & 0\end{pmatrix}$

Row Picture



Column Picture



14. **(Driving a Point Home)** For the following matrix A, find the pivot positions of A and of A^T. Do they have the same pivots? Do they have the same rank?

$$A = \begin{pmatrix}1 & 2 & 3\\-1 & -2 & -3\\4 & 5 & 6\end{pmatrix}$$

In Sympy, the transpose of `A` is `A.T`.

15. **(Internalizing a Concept)** For a certain 4×5 matrix A, the matrix equation

$$Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{has solution set} \quad x = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 4 \\ 1 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Which of the following can you conclude about A, and why?

a) $\mathrm{rank}(A) = 4$ **b)** A has full row rank **c)** A has full column rank

d) $\mathrm{Col}(A) = \mathbf{R}^4$ **e)** $\mathrm{Row}(A) = \mathbf{R}^4$ **f)** $\mathrm{Nul}(A)$ is a line in $\mathbf{R}^5$

g) $Ax = b$ has infinitely many solutions for all $b \in \mathbf{R}^4$

h) A has linearly independent rows **i)** $\mathrm{Nul}(A^T) = \{0\}$

16. **(Internalizing a Concept)** If A is a 5×2 matrix *with full column rank*, which of the following statements must be true about A, and why?

a) $\mathrm{rank}(A) = 5$ **b)** $\mathrm{Col}(A)$ is a plane in $\mathbf{R}^5$ **c)** $\mathrm{Nul}(A) = \{\}$

d) $Ax = b$ has a unique solution for every $b \in \mathbf{R}^5$

e) $Ax = 0$ has a unique solution **f)** $\mathrm{Nul}(A^T)$ is a plane in $\mathbf{R}^5$

g) $\mathrm{Row}(A) = \mathbf{R}^2$ **h)** A has linearly independent columns

17. **(Internalizing a Concept)** If A is a 5×5 matrix, which of the following statements are *equivalent* to the statement "A is invertible"?

a) $\mathrm{rank}(A) = 5$ **b)** A has full row rank **c)** A has full column rank

d) there is a matrix B such that $AB = I_5$ **e)** A has no free columns

f) $Ax = b$ is consistent for every $b \in \mathbf{R}^5$ **g)** $\mathrm{Row}(A) = \mathbf{R}^5$ **h)** $\mathrm{Col}(A) = \mathbf{R}^5$

i) $Ax = (1, 1, 1, 1, 1)$ has exactly one solution

j) $\mathrm{Nul}(A) = \{0\}$ **k)** $\mathrm{Nul}(A^T) = \{0\}$

18. **(Exploration Problem)** In this problem, A and B are two matrices that can be multiplied together.

a) Use HW3#21 to show that $\mathrm{rank}(AB) \le \mathrm{rank}(A)$.

b) Use **a)** and the equalities

$$\mathrm{rank}((AB)^T) = \mathrm{rank}(AB) \qquad \mathrm{rank}(B^T) = \mathrm{rank}(B)$$

to show that $\mathrm{rank}(AB) \le \mathrm{rank}(B)$.

19. **(Exploration Problem)** This problem explains why we only consider *square* matrices when we discuss invertibility.

a) Use Problem 18(a) to show that a tall matrix A (more rows than columns) does not have a right inverse, i.e., there is no matrix B such that $AB = I_m$.

b) Use Problem 18(b) to show that a wide matrix A (more columns than rows) does not have a left inverse, i.e., there is no matrix B such that $BA = I_n$.

20. **(Simpler Descriptions)** Each subspace on the left is equal to one of the subspaces on the right. Which one?

a) $\mathrm{Col}\begin{pmatrix} 1 & 7 & 4 \\ 2 & 0 & 3 \end{pmatrix}$

b) $\mathrm{Nul}\begin{pmatrix} 2 & 4 \\ 1 & 3 \\ 1 & 8 \end{pmatrix}$

c) $\mathrm{Col}\begin{pmatrix} 1 & 7 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

d) $\mathrm{Nul}\begin{pmatrix} 2 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}$

e) $\mathrm{Nul}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}^T$

a) $\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

b) the x-axis in $\mathbf{R}^2$

c) the y-axis in $\mathbf{R}^2$

d) $\mathbf{R}^2$