Math 218D-1: Homework #5

due Wednesday, October 1, at 11:59pm

I know these are long homework assignments. There is a lot of material, and I want to make sure you've had a chance to work through it all before seeing it on an exam—I won't test you on anything you haven't already seen on the homework. Try to articulate what is the point of each problem—there is very little overlap between problems. And as always, your primary reference should be **the course notes.**

1. **(Internalizing Definitions)** Consider the following vectors:

$$u = \begin{pmatrix} .8 \\ .6 \end{pmatrix} \quad v = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a) Compute the lengths $\|u\|, \|v\|$, and $\|w\|$.

b) Use your answers in **a)** to compute the lengths $\|2u\|, \|-v\|$, and $\|3w\|$.

c) Find the unit vectors in the directions of u, v, and w.

d) Check the Schwartz inequalities $|u \cdot v| \leq \|u\| \, \|v\|$ and $|v \cdot w| \leq \|v\| \, \|w\|$.

e) Find the angles between u and v and between v and w.

f) Find the distance from v to w.

Show your work! Check your answers using SymPy, as in:

```
  # Vectors
u = Matrix([.6, .8])
v = Matrix([3, -4])
  # Dot product
pprint(u.dot(v))
  # Length
pprint(u.norm())
```

2. **(Practice with Dot Product Algebra)** If $\|v\| = 8$ and $\|w\| = 5$, what are the smallest and largest possible values of $\|v - w\|$? Justify your answer using the algebra of dot products *and* by drawing a picture.

3. **(Foreshadowing)** If v is a nonzero vector in $\mathbf{R}^3$, give a geometric description of the set S of all vectors $w \in \mathbf{R}^3$ such that $v \cdot w \leq 0$. (If the dot product is negative, what does that say about the angle between the vectors?)

4. **(Practicing a Procedure)** Compute a basis for the orthogonal complement of each of the following spans.

a) $\text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\right\}$ **c)** $\text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}\right\}$

d) $\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ **e)** $\text{Span}\{\}$ **f)** $\text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right\}$

To solve **b)** in SymPy, you could do something like:

```
A = Matrix([[1, 2, 3],
            [4, 5, 6]])
pprint(A.nullspace())
```

5. **(Practicing a Procedure)** Compute a basis for the orthogonal complement of each the following subspaces.

a) $\mathrm{Col}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **b)** $\mathrm{Nul}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **c)** $\mathrm{Row}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$

d) $\mathrm{Nul}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ **e)** $\mathrm{Col}\begin{pmatrix}3&0\\1&-2\\3&1\end{pmatrix}$ **f)** $\mathrm{Col}\begin{pmatrix}1&2&0\\2&1&0\end{pmatrix}$

g) $\left\{x\in\mathbf{R}^3 : \begin{pmatrix}0&6&8\\\frac{1}{2}&0&0\\0&\frac{1}{2}&0\end{pmatrix}x = 2x\right\}$ **h)** $\{(x,y,-x) : x,y\in\mathbf{R}\}$

[**Hint:** solving **a)–d)** requires only one Gauss-Jordan elimination, and **f)** doesn't require any work.]

6. **(Important Example)** The orthogonal complement of each subspace on the left is equal to one of the subspaces on the right. Which one?

a) The x-axis in $\mathbf{R}^3$.	**a)** The x-axis in $\mathbf{R}^3$.
b) The y-axis in $\mathbf{R}^3$.	**b)** The z-axis in $\mathbf{R}^3$.
c) The xy-plane in $\mathbf{R}^3$.	**c)** The xz-plane in $\mathbf{R}^3$.
d) The yz-plane in $\mathbf{R}^3$.	**d)** The yz-plane in $\mathbf{R}^3$.

Do not perform any computations.

7. **(Internalizing a Concept)** We say that subspaces V, W are *orthogonal* if every vector in V is orthogonal to every vector in W.

a) If two lines in $\mathbf{R}^n$ are orthogonal complements, then what is n and why? (Find the big red box.)

b) Give an example of two lines in $\mathbf{R}^n$ (for some n) that are orthogonal to each other but are not orthogonal complements.

8. **(A Useful Shortcut)** If $(a,b,c)\neq(0,0,0)$ then the subspace

$$V=\{(x,y,z)\in\mathbf{R}^3 : ax+by+cz=0\}$$

is a plane in $\mathbf{R}^3$.

a) How do you know a priori that $V^\perp$ is a line, without doing any computations?

b) Explain why $\{(a,b,c)\}$ is a basis for $V^\perp$ without doing any elimination.

The upshot is that (a,b,c) is a *normal vector* to the plane $ax+by+cz=0$.

9. **(Driving a Point Home)** Compute a basis for the orthogonal complement of the plane

$$V = \{(x, y, z) \in \mathbf{R}^3 : x + 3y - 2z = 0\}$$

in two different ways:

a) Express V as a null space, and take the row space.

b) Compute a basis for V, express V as a column space, then find a basis for the left null space.

Verify that your answers to **a)** and **b)** are bases for the *same subspace* (namely, $V^\perp$). Which was easier to compute?

The point here is that the subspace $V^\perp$ does not depend on your choice of description of V, and that some descriptions are more convenient for computations than others.

10. **(Examples Problem)** Construct a matrix A with each of the following properties, or explain why no such matrix exists.

a) The column space contains $(0, 2, 1)$, and the null space contains $(1, -1, 2)$ and $(-1, 3, 2)$.

b) The row space contains $(0, 2, 1)$, and the null space contains $(1, -1, 2)$ and $(-1, 3, 2)$.

c) $Ax = (1, 2, 3)$ is consistent, and $A^T(-1, -1, 2) = 0$.

d) A nonzero 2×2 matrix A such that every row of A is orthogonal to every column.

e) The sum of the columns of A is $(0, 0, 0)$, and the sum of the rows of A is $(1, 1, 1)$.

11. **(Practicing a Procedure)** Express the plane

$$V = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

as the solution set of a system of linear equations. What is the smallest number of equations you need, and why?

12. **(Familiarity with A^TA)** Explain why A has full column rank if and only if A^TA is invertible. (Find the big red box relating A and A^TA.)

13. (Familiarity with A^TA / Foreshadowing) Let Q be an $n \times n$ (*square*) matrix such that $Q^TQ = I_n$. Since Q is square, this means $Q^T = Q^{-1}$.

a) Show that the columns of Q are unit vectors.

b) Show that the columns of Q are orthogonal to each other.

c) Show that $QQ^T = I_n$.

d) Use **a)**, **b)**, and **c)** to show that the *rows* of Q are also orthogonal unit vectors.

e) Find all 2×2 matrices Q such that $Q^TQ = I_2$. (In other words, find all pairs of orthogonal unit vectors in $\mathbf{R}^2$.)

Such a matrix Q is called *orthogonal*.[1]

14. (Practicing a Procedure) For each line L and vector b, compute the orthogonal projection b_L of b onto L using the formula for projection onto a line.

a) $$L = \text{Span}\left\{\begin{pmatrix}2\\3\\4\end{pmatrix}\right\} \qquad b = \begin{pmatrix}5\\4\\9\end{pmatrix}$$

b) $$L = \text{Span}\left\{\begin{pmatrix}-1\\0\\1\\-1\end{pmatrix}\right\} \qquad b = \begin{pmatrix}4\\2\\2\\1\end{pmatrix}$$

Check your answers in SymPy, as in:

```
v = Matrix([2, 3, 4])
b = Matrix([5, 4, 9])
bV = b.dot(v)/v.dot(v)*v
pprint(bV)
```

[1] I am not responsible for this terminology.

15. **(Practicing a Procedure)** For each subspace V and vector b, compute the orthogonal projection b_V of b onto V by solving a normal equation $A^T Ax = A^T b$. Then compute the orthogonal decomposition $b = b_V + b_{V^\perp}$, and find the distance from b to V.

a) $$V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

b) $$V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$$

c) $$V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$$

Compute $A^T A$ by hand, but remember that you're just computing column dot products! Check your answers using SymPy, as in:

```
A = Matrix([[1, 1],
            [1, 0],
            [0, 2]])
b = Matrix([1, 4, 3])
  # This will only work when A has FCR!
x = (A.T*A).solve(A.T*b)
bV = A*x
bVperp = b - bV
pprint(bV)
pprint(bVperp)
pprint(bVperp.norm())
```

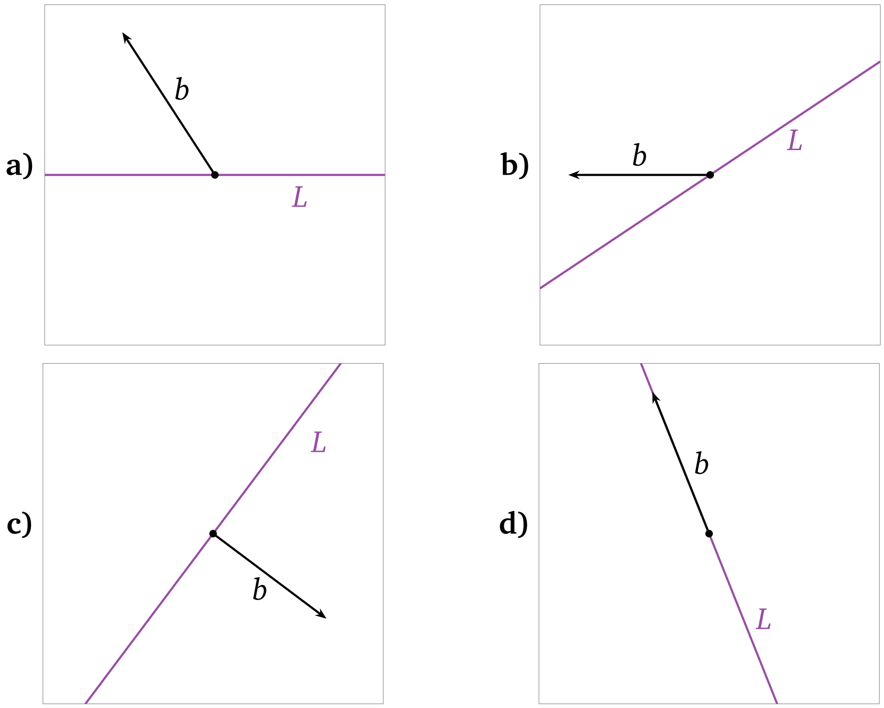
16. **(Practicing a Procedure)** For each subspace V and vector b, compute the orthogonal projection b_V of b onto V by first computing $b_{V^\perp}$.

a) $$V = \mathrm{Nul}\begin{pmatrix} 1 & -1 & 4 \\ 2 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ -9 \\ 5 \end{pmatrix}$$

b) $$V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 5 \\ 2 \\ 5 \\ 3 \end{pmatrix}$$

17. **(Internalizing a Concept)** Let A be a matrix with n columns and let b be a vector in $\mathbf{R}^n$. Explain why b can be expressed as the sum of a vector in Nul(A) and a vector in Row(A). How would you compute these vectors?

18. (Picture Problem) In each case, *draw* the orthogonal projections b_L and $b_{L^\perp}$ *without doing any computations*.



19. (Driving a Point Home) Consider the line

$$L = \mathrm{Nul}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Compute the orthogonal projection of $b = (1, 6, 9)$ onto L in two different ways, as follows:

a) by first computing $b_{L^\perp}$ (solving a normal equation);

b) by finding a basis for L and using projection onto a line.

The point is that these two *different descriptions* of L must give the *same answer* for the projection b_L.

20. **(Important Example)** It is easy to project onto coordinate axes and coordinate planes.

a) Find the orthogonal projection of (a, b, c) onto the x-axis.

b) Find the orthogonal projection of (a, b, c) onto the y-axis.

c) Find the orthogonal projection of (a, b, c) onto the xy-plane.

d) Find the orthogonal projection of (a, b, c) onto the yz-plane.

In each case, explain your answer, but *do not do any computations*. Instead, eyeball a vector x on the subspace such that $(a, b, c) - x$ is orthogonal to the subspace. See Problem 6.

21. **(Exploration Problem)**

a) Let $v, w \in \mathbf{R}^n$. Show that

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

if $v \perp w$. (Do dot product algebra.)

b) Let V be a subspace of $\mathbf{R}^n$, let $b \in \mathbf{R}^n$, and let $v \in V$. Explain why $b - b_V$ is orthogonal to $b_V - v$, then apply **a)** to the sum $b - v = (b - b_V) + (b_V - v)$ to show that

$$\|b - v\|^2 = \|b - b_V\|^2 + \|b_V - v\|^2.$$

Use this to prove that

$$\|b - v\|^2 \geq \|b - b_V\|^2,$$

i.e., that b_V really is the closest vector in V to b.

c) Let V be a subspace of $\mathbf{R}^n$ and let $b \in \mathbf{R}^n$. Apply **a)** to the orthogonal decomposition of b to show that $\|b_V\| \leq \|b\|$, with equality if and only if $b \in V$.

In other words, orthogonal projection can only make a vector *shorter.*