Math 218D-1: Homework #6

due Wednesday, October 8, at 11:59pm

> As always, you should use SymPy on the Sage cell to compute orthogonal projections and projection matrices once you are comfortable computing them by hand. If your answers have fractions with large denominators, it's probably because I didn't try very hard to make the numbers work out nicely because I was expecting you to ask the computer to do the computation—not because you did the problem wrong.

1. **(Internalizing a Concept)** For each subspace V, find the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1, 2, -1)$ with respect to V *without doing any computations.*

$$\textbf{a) } V = \operatorname{Span}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix}\right\} \qquad \textbf{b) } V = \operatorname{Col}\begin{pmatrix}2 & 0\\-1 & 1\\0 & 2\end{pmatrix}$$

$$\textbf{c) } V = \mathbf{R}^3 \qquad \textbf{d) } V = \{0\}$$

2. **(Practicing a Procedure)** For each vector v, compute the projection matrix onto $V = \operatorname{Span}\{v\}$ using the formula $P_V = vv^T/v \cdot v$.

$$\textbf{a) } v = \begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \textbf{b) } v = \begin{pmatrix}3\\0\\4\\-1\end{pmatrix} \qquad \textbf{c) } v = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \quad (\text{in } \mathbf{R}^n)$$

Check your answers for **a)** and **b)** with SymPy, as in:

```
v = Matrix([1, 2, 3])
pprint(v*v.T/v.dot(v))
```

3. **(Practicing a Procedure)** For each subspace V, compute the projection matrix P_V. *Show your work!*

a) $V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ **b)** $V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix}$ **c)** $V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

Check your answers with SymPy, as in:

```
B = Matrix([[1, 1],
            [1, 0],
            [0, 2]])
PV = B*(B.T*B).inv()*B.T
pprint(PV)
```

Use SymPy to verify that $P_V^2 = P_V$ and $P_V^T = P_V$ by computing `PV*PV-PV` and `PV.T-PV`.

4. **(Practicing a Procedure)** For each subspace V, compute the projection matrix P_V by first computing $P_{V^\perp}$.

a) $V = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ **b)** $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

5. For each subspace V, compute the projection matrix P_V.

a) $\{(x, y, -x) : x, y \in \mathbf{R}\}$.

b) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

c) **(Challenge)** The intersection of the plane $x - y + 2z = 0$ with the plane spanned by $(1, 1, 1)$ and $(1, 2, -1)$.

6. **(Driving a Point Home)** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 3 \end{pmatrix},$$

and let $V = \mathrm{Col}(A)$.

a) Compute P_V using the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute a basis $\{v_1, v_2\}$ for $V^\perp = \mathrm{Nul}(A^T)$.

c) Let B be the matrix with columns v_1, v_2, and compute $P_{V^\perp}$ using the formula $B(B^TB)^{-1}B^T$.

d) Verify that your answers to **a)** and **c)** sum to I_4.

This illustrates the fact that once you've computed P_V, there's no need to compute $P_{V^\perp}$ separately. It's a lot of extra work!

7. **(Internalizing a Concept)** A certain subspace V of $\mathbf{R}^3$ has the projection matrix

$$P_V = \frac{1}{14}\begin{pmatrix} 10 & -6 & -2 \\ -6 & 5 & -3 \\ -2 & -3 & 13 \end{pmatrix}.$$

In this problem, *do not do any computations whatsoever.*

a) Find the orthogonal projection of $(0,1,0)$ onto V.

b) Find a basis for V, and determine $\dim(V)$.

c) Find a basis for $V^\perp$.

8. **(Internalizing a Concept)** Let V be a subspace of $\mathbf{R}^n$, with orthogonal complement $V^\perp$. Show that $P_V P_{V^\perp} = 0$ in two ways:

a) Do matrix algebra, using $P_{V^\perp} = I_n - P_V$ and $P_V^2 = P_V$.

b) Think geometrically: where does $P_V P_{V^\perp}$ send a vector b? How does this tell you what $P_V P_{V^\perp}$ is?

9. **(Important Example)** It is easy to compute the projection matrix onto coordinate axes and coordinate planes.

a) Find the projection matrix onto the x-axis in $\mathbf{R}^3$.

b) Find the projection matrix onto the y-axis in $\mathbf{R}^3$.

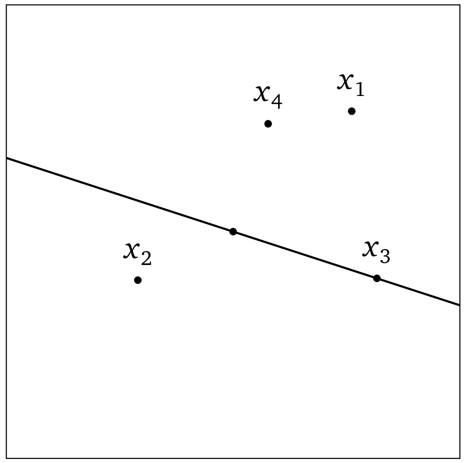
c) Find the projection matrix onto the xz-plane in $\mathbf{R}^3$.

d) What do you think is the matrix for projection onto $\mathrm{Span}\{e_2, e_4\}$ in $\mathbf{R}^4$? Don't do any computations; just continue the pattern.

10. Let V be a subspace of $\mathbf{R}^n$. The matrix for *reflection over* V is

$$R_V = I_n - 2P_{V^\perp}.$$

a) Suppose that V is the line in the picture. Draw the vectors $R_V x_1, R_V x_2, R_V x_3$, and $R_V x_4$ as points in the plane. (Distribute the product $(I_2 - 2P_{V^\perp})x_i$ and draw the summands.)



b) In general, show that $R_V^T = R_V$ and that $R_V^2 = I_n$. (Do matrix algebra.)

11. **(Examples Problem)** In each part, find an example or explain why no such example exists.

a) A 3×3 projection matrix of rank 3.

b) A 3×2 matrix A of rank 2 such that the orthogonal projection of $(1,1,1)$ onto $\mathrm{Col}(A)$ is zero.

c) A 4×4 projection matrix whose null space is equal to its column space.

d) A 4×4 projection matrix whose row space is equal to its column space.

e) Nonzero vectors v and w in $\mathbf{R}^3$ such that $P_V P_W = 0$, where $V = \mathrm{Span}\{v\}$ and $W = \mathrm{Span}\{w\}$.

f) A plane V in $\mathbf{R}^3$ such that $\mathrm{rank}(P_V) = 1$.

12. **(Practicing a Procedure)** Find all least-squares solutions $\widehat{x}$ of each of the following systems of equations $Ax = b$. Then compute the projection $b_V = A\widehat{x}$ of b onto $V = \mathrm{Col}(A)$ and the error $\|b - A\widehat{x}\|$. *Show your work.*

a) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} x = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} x = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$ **d)** $\begin{pmatrix} 3 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix} x = \begin{pmatrix} 9 \\ 7 \\ 7 \end{pmatrix}$

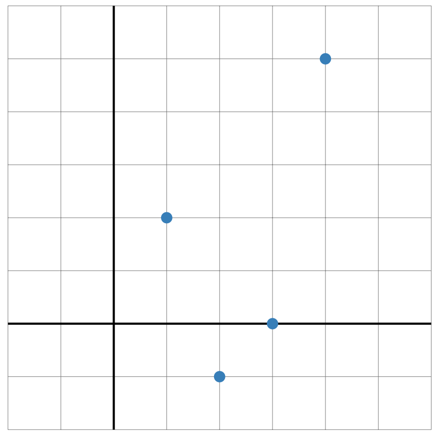
Check your answers with the Sage cell, as in:

```
A = Matrix([[1, 1],
            [1, 0],
            [0, 2]])
b = Matrix([1, 4, 3])
  # If A has FCR, this will work:
pprint((A.T*A).solve(A.T*b))
  # Otherwise, this will find the parametric form:
pprint((A.T*A).gauss_jordan_solve(A.T*b))
  # Or, you can form the augmented matrix (ATA | ATb):
B = (A.T*A).row_join(A.T*b)
pprint(B.rref(pivots=False))
```

13. **(Practicing a Procedure)** Consider the data points

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad p_4 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

a) Find the best-fit line $y = Cx + D$ (in the sense of least squares) through these four points, and draw it on the grid below.



b) For each data point $p_i = (a_i, b_i)$, draw the error bar from $(a_i, y(a_i))$ to (a_i, b_i).

c) What is the minimum value of $\sum_{i=1}^{4}(b_i - y(a_i))^2$? How do you know?

d) Verify that the vector

$$\big(2 - y(1),\ -1 - y(2),\ 0 - y(3),\ 5 - y(4)\big)$$

is orthogonal to $(1, 2, 3, 4)$ and $(1, 1, 1, 1)$, and explain why this is necessary.

e) Find the best-fit *horizontal* line $y = D$ through these four points. Verify that D is the average of the y-values of the data points p_1, p_2, p_3, p_4.

14. **(Practicing a Procedure)** Consider the following data points:

$$p_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad p_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \qquad p_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

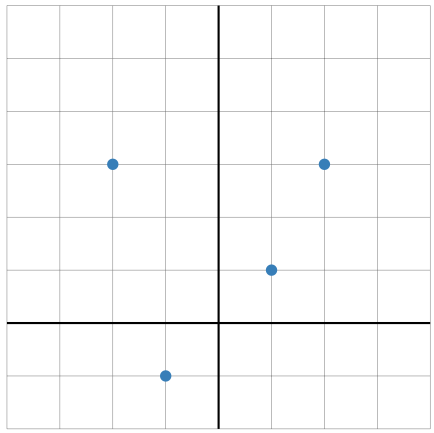
a) Find the best-fit plane $z = Cx + Dy + E$ through these four points.

b) Interpret the minimized quantity in the situation of this problem.

15. **(Practicing a Procedure)** Consider the data points

$$p_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad p_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad p_4 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

a) Find the best-fit parabola $y = Cx^2 + Dx + E$ through these four points, and draw it on the grid below.



b) For each data point $p_i = (a_i, b_i)$, draw the error bar from $(a_i, y(a_i))$ to (a_i, b_i).

c) What geometric quantity did you minimize?

16. **(Practicing a Procedure)** Consider the data points $p_1, \ldots, p_8$:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}, \begin{pmatrix} 2.5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -.5 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2.5 \\ 2 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 3.5 \end{pmatrix}.$$

a) Find the best-fit *ellipse*

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

through these data points.

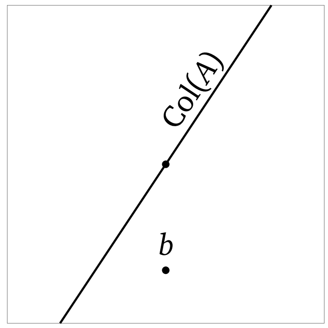
b) Interpret the minimized quantity in the situation of this problem.
[**Hint:** you can't see it on the graph above, but you can see it on this demo.]

You'll definitely want to use SymPy to do this problem. Decimal answers are okay.

Remark: Carl Friedrich Gauss (1777–1865), arguably the greatest mathematician since antiquity, kept food on the table by doing astronomical calculations. He invented much of the linear algebra you are learning in order to compute the trajectories of celestial bodies. Essentially performing the calculations in this problem, he correctly predicted the (elliptical) orbit of the asteroid Ceres as it passed behind the sun in 1801.

17. **(Internalizing a Concept)** Suppose that $\widehat{x}$ is a vector such that $A\widehat{x} = (1, 1, -1, -1)$. Explain why $\widehat{x}$ is not a least-squares solution of $Ax = (1, 1, 1, 1)$.

18. **(Picture Problem)** The column space of a certain 2×3 matrix A is drawn below, along with a vector $b \in \mathbf{R}^2$. Draw the set of all vectors b' such that $Ax = b$ and $Ax = b'$ have the same least squares solutions.



19. **(Examples Problem)** In each case, find an example or explain why none exists.

a) A 2×2 matrix A and a vector $b \in \mathbf{R}^2$ such that $Ax = b$ has exactly one least-squares solution.

b) A 2×2 matrix A and a vector $b \in \mathbf{R}^2$ such that $Ax = b$ has infinitely many least-squares solutions.

c) A 2×2 matrix A and a vector $b \in \mathbf{R}^2$ such that $Ax = b$ does not have any least squares solutions.

d) A 2×2 matrix A and a vector $b \in \mathbf{R}^2$ such that *every* $x \in \mathbf{R}^2$ is a least-squares solution of $Ax = b$.

e) A 2×2 matrix A and a vector $b \in \mathbf{R}^2$ such that $Ax = b$ has no ordinary solutions and has exactly one least-squares solution.

f) A 2×2 matrix A and a vector $b \in \mathbf{R}^2$ such that $\mathrm{Nul}(A) = \mathrm{Span}\left\{\binom{1}{1}\right\}$ and the set of least squares solutions of $Ax = b$ is $\binom{1}{0} + \mathrm{Span}\left\{\binom{1}{-1}\right\}$.