Math 218D-1: Homework #7

due Wednesday, October 15, at 11:59pm

> Once you are comfortable with finding least-squares solutions by hand, please perform those computations in SymPy! The same thing goes for the Gram–Schmidt process and QR decompositions.

1. **(Internalizing a Definition)** For each set of vectors, decide if they are orthogonal, orthonormal, or neither; then compute $Q^T Q$ *by hand*, where Q is the matrix with the vectors as columns.

a) $\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}$ **b)** $\left\{\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\right\}$

c) $\left\{\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}, \frac{1}{3}\begin{pmatrix}-2\\1\\2\end{pmatrix}\right\}$ **d)** $\left\{\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, \frac{1}{\sqrt{18}}\begin{pmatrix}-4\\1\\1\end{pmatrix}\right\}$

2. **(Practicing a Procedure)** The following subspaces V are given as the span of an *orthogonal* set of vectors. For each subspace V and vector b, compute the orthogonal projection b_V using the *projection formula*, and compute the projection matrix P_V using the *outer product formula*.

a) $V = \text{Span}\left\{\begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\-2\\1\end{pmatrix}\right\}$ $\qquad b = \begin{pmatrix}3\\8\\1\end{pmatrix}$

b) $V = \text{Span}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\}$ $\qquad b = \begin{pmatrix}0\\3\\6\end{pmatrix}$

c) $V = \text{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\}$ $\qquad b = \begin{pmatrix}4\\2\\-4\\2\end{pmatrix}$

d) $V = \text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ $\qquad b = \begin{pmatrix}7\\-3\\2\end{pmatrix}$

e) $V = \text{Span}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}-4\\1\\1\end{pmatrix}\right\}$ $\qquad b = \begin{pmatrix}9\\-2\\3\end{pmatrix}$

3. Suppose that $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis of $\mathbf{R}^n$. Use the outer product formula to explain why

$$I_n = u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T.$$

4. **(Examples Problem)** In each case, find an example or explain why none exists.

a) A matrix Q such that Q^TQ is an identity matrix but QQ^T is not.

b) A set of vectors $\{v_1, v_2, v_3\}$ that is *orthogonal* and *linearly dependent*.

c) An orthonormal basis for the plane $x + y + z = 0$.

5. (Practicing a Procedure) Use the Gram–Schmidt process to find orthogonal bases of the following subspaces.

a) $\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix}1\\-1\\2\\-2\end{pmatrix}, \begin{pmatrix}1\\-3\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\4\\-1\end{pmatrix}\right\}$

c) $\text{Span}\left\{\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ **d)** $\text{Nul}\begin{pmatrix}1 & 2 & -1 & 4\\3 & 6 & -3 & 12\end{pmatrix}$

Check your answers with SymPy, as in:

```
GramSchmidt([Matrix([1, 1, 0]),
             Matrix([1, 0, 2])])
```

6. (Understanding Gram–Schmidt) Consider the plane

$$V = \text{Span}\left\{\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\\1\end{pmatrix}\right\}$$

and the vector

$$v = \begin{pmatrix}4\\-1\\3\\3\end{pmatrix} = 2\begin{pmatrix}1\\-1\\2\\1\end{pmatrix} + \begin{pmatrix}2\\1\\-1\\1\end{pmatrix} \in V.$$

Find all vectors **contained in** V that are orthogonal to v.
[**Hint:** apply Gram–Schmidt to a set containing v.]

7. (Understanding Gram–Schmidt) Consider the 3-dimensional subspace $V = \text{Col}(A)$, where

$$A = \begin{pmatrix}1 & 1 & 2\\-1 & -3 & 2\\2 & 2 & 4\\-2 & -1 & -1\end{pmatrix}.$$

Find an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathbf{R}^4$ such that $\{u_1, u_2, u_3\}$ is a basis for V. Your answer should be exact, in terms of square roots.

8. **(Practicing a Procedure)** For each of the following matrices A and vectors b, find the QR decomposition of A, and find the least-squares solution of $Ax = b$ by substitution in $R\widehat{x} = Q^T b$. Your answers should be exact, in terms of square roots.

$$\textbf{a)}\ A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \textbf{b)}\ A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -3 & 2 \\ 2 & 2 & 4 \\ -2 & -1 & -1 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Check your answers with SymPy, as in:

```
A = Matrix([[1, 1],
            [1, 0],
            [0, 2]])
Q, R = A.QRdecomposition()
pprint(Q)
pprint(R)
b = Matrix([1, 1, 1])
pprint(R.upper_triangular_solve(Q.T*b))
```

9. **(Practicing a Procedure)** In this problem, we use a QR decomposition to *quickly* compute the best-fit parabola with specified y-values at $x = -2, -1, 1, 2$, as in HW6#15.

a) Find the matrix A such that the least squares solution of

$$A \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

gives the coefficients of the parabola $y = Cx^2 + Dx + E$ that best fits the data points

$$\begin{pmatrix} -2 \\ b_1 \end{pmatrix},\ \begin{pmatrix} -1 \\ b_2 \end{pmatrix},\ \begin{pmatrix} 1 \\ b_3 \end{pmatrix},\ \begin{pmatrix} 2 \\ b_4 \end{pmatrix}.$$

(Presumably you did this in HW6#15.)

b) Compute the QR decomposition of A.

c) Find the best-fit parabola through the points $\binom{-2}{3}$, $\binom{-1}{-1}$, $\binom{1}{1}$, $\binom{2}{3}$ by substitution in $R\widehat{x} = Q^T b$. You should get the same answer as in HW6#15.

Note that we can now repeat part **c)** with new y-values in $O(n^2)$ time.

10. **(Internalizing a Definition)** Recall that an *orthogonal* matrix is a *square* matrix Q with ortho*normal* columns, and that orthonormal columns means $Q^T Q = I_n$.

a) If Q is an orthogonal matrix, show that Q^{-1} is orthogonal.

b) If Q_1 and Q_2 are orthogonal matrices of the same size, show that $Q_1 Q_2$ is orthogonal.

11. **(True-False)** Decide if each statement is true or false. If it is true, explain why; if it is false, provide a counterexample.

a) A matrix with orthogonal columns has full row rank.

b) If $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors, then it is orthogonal.

c) If $\{v_1, v_2\}$ is a basis for a plane V, then for any vector b,

$$b_V = \frac{b \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{b \cdot v_2}{v_2 \cdot v_2} v_2.$$

d) If Q has orthonormal columns, then the distance from x to y equals the distance from Qx to Qy.

e) If $A = QR$ is a QR decomposition of a matrix A, then the rows of Q form an orthonormal basis for $\mathrm{Row}(A)$.

12. **(Practicing a Procedure)** Compute the determinants of the following matrices *using Gaussian elimination*.

a) $\begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix}$

c) $\begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix}$ **d)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

Check your answers with SymPy, as in:

```
A = Matrix([[-2, 1],
            [ 1, 3]])
pprint(A.det())
```

13. **(Internalizing a Definition)**

a) Find $\det(E)$ when E is an elementary matrix. (This will depend on which kind of row operation the matrix corresponds to.)

b) Verify that $\det(E) = \det(E^T)$ when E is an elementary matrix.
[**Hint:** for row swaps, use HW2#22(b).]

c) If A is a matrix and E is an elementary matrix, use **a)** to show that $\det(EA) = \det(E)\det(A)$ directly from the definition of determinants (not using multiplicativity).
[**Hint:** how is EA related to A?]

14. (Internalizing a Definition) Let A be the $n \times n$ matrix with entries $1, 2, 3, \ldots, n^2$, ordered by rows first. For instance, here is the matrix A for $n = 2$ and $n = 3$:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Find $\det(A)$ for any value of $n \geq 2$.

15. (Internalizing a Concept) A matrix A has the $PA = LU$ factorization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A = L \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

What is $\det(A)$?

16. (Internalizing a Concept) Suppose that

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10.$$

Find the determinants of the following matrices.

a) $\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$ **b)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g+2d & h+2e & i+2f \end{pmatrix}$ **c)** $\begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix}$

d) $\begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$ **e)** $\begin{pmatrix} a & b & c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix}$ **f)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

g) $2\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **h)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **i)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}$

j) $-\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **k)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^3$ **l)** $\begin{pmatrix} a & b+2c & c \\ d & e+2f & f \\ g & h+2i & i \end{pmatrix}$

17. (Synthesizing New and Old Concepts) Recall that an *orthogonal matrix* is a square matrix with orthonormal columns, or equivalently, a square matrix Q such that $Q^T Q = I_n$. Prove that every orthogonal matrix has determinant ± 1.

18. **(Synthesizing New and Old Concepts)** Let V be a subspace of $\mathbf{R}^n$ and let P_V be the projection matrix onto V.

a) Find $\det(P_V)$ when $V \neq \mathbf{R}^n$.

b) Find $\det(P_V)$ when $V = \mathbf{R}^n$.

19. **(Exploration Problem)** Let A be any invertible matrix.

a) Explain why A can be expressed as a product of elementary matrices:

$$A = E_1 E_2 \ldots E_r.$$

b) Use Problem 13(c) to prove that $\det(A) = \det(E_1)\det(E_2)\cdots\det(E_r)$ directly from the definition of the determinant (without using multiplicativity).

c) Take transposes and use Problem 13(b) and **b)** (as applied to A^T) to prove that $\det(A) = \det(A^T)$.

[**Hint:** the transpose of an elementary matrix is an elementary matrix.]

(The proof that $\det(AB) = \det(A)\det(B)$ follows a similar strategy.)

20. **(Exploration Problem)** Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$.

a) Show that if $\{v_1, v_2, \ldots, v_n\}$ is orthogonal then $|\det(A)| = \|v_1\|\,\|v_2\|\cdots\|v_n\|$.

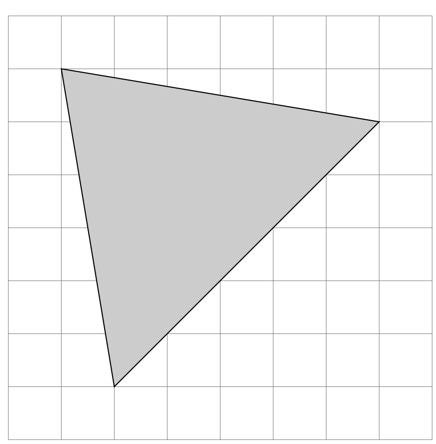
[**Hint:** Compute A^TA and its determinant.]

b) Suppose that A is invertible. Show that $|\det(A)| \leq \|v_1\|\,\|v_2\|\cdots\|v_n\|$, with equality if and only if the set $\{v_1, v_2, \ldots, v_n\}$ is orthogonal.

[**Hint:** Use HW5#21(c) and the QR decomposition of A. In $A = QR$, you know what $|\det(Q)|$ and $\det(R)$ are…]

In other words, among matrices with the same column lengths, the determinant is *maximized* when the columns are *orthogonal*.

21. **(Picture Problem)** Compute the area of the triangle pictured below using a 2×2 determinant. (The grid marks are one unit apart.)



22. **(True-False)** Decide if each statement is true or false. If it is true, explain why; if it is false, provide a counterexample. You can assume that all matrix operations that appear below are defined.

a) $\det(A+B) = \det(A) + \det(B)$.

b) $\det(ABC^{-1}) = \dfrac{\det(A)\det(B)}{\det(C)}$.

c) $\det(AB) = \det(BA)$.

d) $\det(3A) = 3\det(A)$.

e) If A^5 is invertible then A is invertible.

f) The determinant of A is the product of its diagonal entries.

g) If the columns of A are linearly dependent, then $\det(A) = 0$.

h) If A is a 3×3 matrix with determinant zero, then two of the columns of A are scalar multiples of each other.