Math 218D-1: Homework #8

due Wednesday, October 22, at 11:59pm

1. **(Practicing a Procedure)** Consider these matrices from HW7#12:

$$A = \begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

a) Compute $\det(A)$ using Sarrus' Scheme.

b) Compute $\det(B)$ by expanding cofactors along a row.

c) Compute $\det(C)$ by expanding cofactors along a column.

You should get the same answers as you got in HW7#12.

d) Now try to compute $\det(B)$ using Sarrus' scheme, by summing the products of the forward diagonals and subtracting the products of the backward diagonals. Did you get the determinant?

2. **(Practicing a Procedure)** Compute

$$\det\left[\begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} - \lambda I_3\right]$$

where λ is an unknown real number. Your answer will be a function of λ. *Show your work.*

Check your answer with SymPy:

```
  # We'll use x as a variable instead of lambda.
  # Tell SymPy that x is a symbol:
x = symbols('x')
A = Matrix([[-3,  3, 2],
            [ 3,  0, 0],
            [-9, 18, 7]])
  # eye(3) is the 3x3 identity matrix
pprint((A-x*eye(3)).det())
```

3. **(Internalizing a Concept)** Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

a) Find $\det(A)$.

b) Compute the cofactor matrix C of A.

c) Compute AC^T.

What is the relationship between C^T, $\det(A)$, and A^{-1}?

4. Consider the $n \times n$ matrix F_n with 1's on the diagonal, 1's in the entries immediately below the diagonal, and -1's in the entries immediately above the diagonal:

$$F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \cdots.$$

a) Show that $\det(F_2) = 2$ and $\det(F_3) = 3$.

b) Expand in cofactors to show that $\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})$.

c) Compute $\det(F_4), \det(F_5), \det(F_6), \det(F_7)$ using **b)**.

This shows that $\det(F_n)$ is the nth *Fibonacci number*. (The sequence usually starts with $1, 1, 2, 3, \ldots$, so our $\det(F_n)$ is the usual $n+1$st Fibonacci number.)

There will be more on Fibonacci numbers when we learn to solve difference equations!

5. **(Exploration Problem)** Let A be an $n \times n$ invertible matrix with integer (whole number) entries.

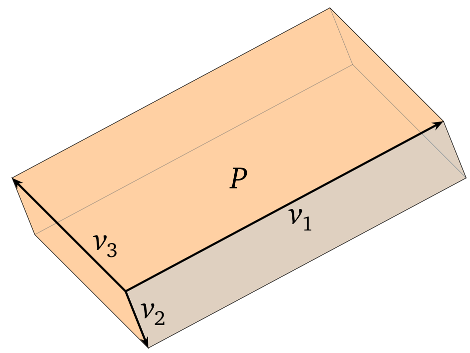
a) Explain why $\det(A)$ is an integer.

b) If $\det(A) = \pm 1$, show that A^{-1} has integer entries.

c) If A^{-1} has integer entries, show that $\det(A) = \pm 1$.
[**Hint:** What is $\det(A)\det(A^{-1})$?]

6. **(Practicing a Procedure)** Consider the paralellepiped P in $\mathbf{R}^3$ spanned by

$$v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$



a) Compute the volume of P using the triple product $(v_1 \times v_2) \cdot v_3$.

b) Explain why the face of P spanned by v_i and v_j has area equal to $\|v_i \times v_j\|$. (Use the formula $\|v_i \times v_j\| = \|v_i\|\,\|v_j\|\sin(\theta)$.)

c) Compute the area of each face of P using cross products.

7. (Internalizing a Concept)

a) Let $v = \binom{a}{b}$ and $w = \binom{c}{d}$ be vectors in the plane, and let $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. By taking the cross product of $(a, b, 0)$ and $(c, d, 0)$ and using the right-hand rule, explain when $\det(A)$ is positive or negative in geometric terms.

b) Taking the triple product

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right] \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},$$

and using the right-hand rule, explain when the determinant of a 3×3 matrix is positive or negative in geometric terms. (See HW5#3.)

8. (Internalizing a Concept) Let V be a subspace of $\mathbf{R}^n$. Recall from HW6#10 that the matrix for *reflection over* V is

$$R_V = I_n - 2P_{V^\perp}.$$

a) Use HW6#10(b) to show that $\det(R_V) = \pm 1$.

b) Let $V = \{(x, y, z) \in \mathbf{R}^3 : x + y + z = 0\}$. Compute R_V, and use Problem 7(b) to determine the sign of $\det(R_V)$.

9. (Practicing a Procedure) Use a cross product to find an implicit equation for the plane

$$V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\right\}.$$