Math 218D-1: Homework #9

due Wednesday, October 29, at 11:59pm

1. **(Internalizing a Definition)** For each matrix A and each vector v, decide if v is an eigenvector of A, and if so, find the eigenvalue λ.

a) $\begin{pmatrix} -20 & 42 & 58 \\ 1 & -1 & -3 \\ -1 & 18 & 26 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 3 & 0 \\ -5 & 4 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} -7 & 32 & -76 \\ 7 & -22 & 59 \\ 3 & -11 & 28 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

e) $\begin{pmatrix} -3 & 2 & -3 \\ 3 & -3 & -2 \\ -4 & 2 & -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

[**Hint:** Compute Av.]

2. **(Practicing a Procedure)** For each matrix A and each number λ, decide if λ is an eigenvalue of A; if so, find a basis for the λ-eigenspace of A.

a) $\begin{pmatrix} -5 & -14 \\ 3 & 8 \end{pmatrix}, \ \lambda = 1$ **b)** $\begin{pmatrix} -5 & -14 \\ 3 & 8 \end{pmatrix}, \ \lambda = -1$

c) $\begin{pmatrix} -2 & 3 & -15 \\ 5 & -7 & 31 \\ 2 & -3 & 13 \end{pmatrix}, \ \lambda = 3$ **d)** $\begin{pmatrix} -2 & 3 & -15 \\ 5 & -7 & 31 \\ 2 & -3 & 13 \end{pmatrix}, \ \lambda = 2$

e) $\begin{pmatrix} 3 & 1 & -2 \\ -2 & 0 & 4 \\ -1 & -1 & 4 \end{pmatrix}, \ \lambda = 2$ **f)** $\begin{pmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \\ -1 & -1 & 2 \end{pmatrix}, \ \lambda = 0$

g) $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \ \lambda = 7$ **h)** $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda = 0$

i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \lambda = 0$

You can find eigenvectors using SymPy:

```
A = Matrix([[-5, -14],
            [ 3,   8]])
pprint((A-1*eye(2)).nullspace())
```

3. Suppose that A is an invertible $n \times n$ matrix such that $Av = \lambda v$ for some $v \neq 0$. Show that v is an eigenvector of each of the following matrices, and find the eigenvalue.

a) A^{-1} **b)** $A + 3I_n$ **c)** A^3 **d)** $3A$.

4. **(Ubiquitous Shortcut)** Here is a handy trick for computing eigenvectors of a 2×2 matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with eigenvalue λ. Show that

$$(A - \lambda I_2)\begin{pmatrix} -b \\ a - \lambda \end{pmatrix} = 0 \quad \text{and} \quad (A - \lambda I_2)\begin{pmatrix} d - \lambda \\ -c \end{pmatrix} = 0.$$

Hence $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ and $\begin{pmatrix} d-\lambda \\ -c \end{pmatrix}$ are λ-eigenvectors of A if they are nonzero. If both are equal to zero, what are the eigenvectors of A?

5. **(Exploration Problem)**

a) Show that A and A^T have the same characteristic polynomial. In particular, they have the same eigenvalues.

b) Give an example of a 2×2 matrix A such that A and A^T do not share any eigenvectors.

c) A *stochastic matrix* is a matrix with nonnegative entries such that the entries in each column sum to 1. Explain why 1 is an eigenvalue of a stochastic matrix. [**Hint:** show that $(1, 1, \dots, 1)$ is an eigenvector of A^T.]

6. **(Important Example)**

a) Find all eigenvalues of the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 3 & 4 \\ 0 & 3 & -1 & -2 & -5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

b) Explain how to find the eigenvalues of any triangular matrix without doing any work.

7. **(Important Example)** Recall that an *orthogonal matrix* is a square matrix with orthonormal columns. Prove that any (real) eigenvalue of an orthogonal matrix Q is ± 1.

[**Hint:** What do we know about orthogonal matrices and lengths?]

8. **(Internalizing a Concept)** Let v be an eigenvector of a square matrix A. Show that $v \in \mathrm{Col}(A)$ or $v \in \mathrm{Nul}(A)$.

9. **(Examples Problem)** Give an example of each of the following, or explain why no example exists.

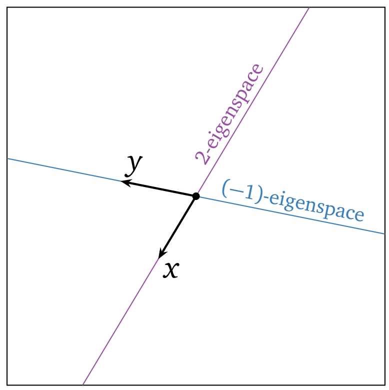
a) An invertible matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda$.

b) A nonzero 2×2 matrix with eigenvalue 0.

c) A 2×2 projection matrix that does not have eigenvalue 1.

d) A 2×2 orthogonal matrix with no real eigenvalues. (We saw one in L16.)

10. **(Picture Problem)** A certain 2×2 matrix A has eigenspaces indicated in the picture. Draw Ax and Ay.



11. **(Practicing a Procedure)** For each 2×2 matrix A, **i)** compute the characteristic polynomial using the formula $p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A)$. Use this to **ii)** find all real eigenvalues, and **iii)** find a basis for each eigenspace, using Problem 4 when applicable. **iv)** Draw and label each eigenspace. **v)** Is the matrix diagonalizable (over the real numbers)?

a) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

12. Let V be the plane $x + y + z = 0$, and let $R_V = I_3 - 2P_{V^\perp}$ be the reflection matrix over V, as in HW8#8. Find all eigenvectors of R_V geometrically, without doing any computations. Is R_V diagonalizable?

13. **(Practicing a Procedure)** For each matrix A, **i)** find all real eigenvalues of A, and **ii)** find a basis for each eigenspace. **iii)** Is the matrix diagonalizable (over the real numbers)?

a) $\begin{pmatrix} -1 & 7 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 7 & 12 & 12 \\ -8 & -13 & -12 \\ 4 & 6 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 6 & 2 & 3 \\ -14 & -7 & -12 \\ 1 & 2 & 4 \end{pmatrix}$

Optional (if you want more practice):

d) $\begin{pmatrix} -11 & -54 & 10 \\ -2 & -7 & 2 \\ -21 & -90 & 20 \end{pmatrix}$ **e)** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

f) $\begin{pmatrix} 13 & 18 & -18 \\ -12 & -17 & 18 \\ -4 & -6 & 7 \end{pmatrix}$ **g)** $\begin{pmatrix} -10 & 28 & -18 & -76 \\ -1 & 9 & -6 & -2 \\ 4 & -8 & 7 & 26 \\ 0 & 2 & -2 & 4 \end{pmatrix}$

You'll want to compute the roots of the characteristic polynomials in SymPy:

```
pprint(roots(-x**3 + 13/4*x + 3/2, multiple=True))
```

Once you're comfortable with this procedure, you can do the whole thing with SymPy:

```
A = Matrix([[ 7,  12,  12],
            [-8, -13, -12],
            [ 4,   6,   5]])
pprint(A.eigenvects())
  # The output is a list of triples
  #   [(eigenvalue, multiplicity, eigenspace basis), ...]
```

Of course, SymPy might find a different eigenspace basis than you.

14. **(Practicing a Procedure)** For each matrix A and initial state v_0, **i)** solve the difference equation $v_{k+1} = Av_k$ (i.e. compute $v_k = A^k v_0$) using diagonalization, and **ii)** describe the behavior of v_k as $k \to \infty$.

a) $$A = \begin{pmatrix} 12 & 35 \\ -10/3 & -29/3 \end{pmatrix} \quad v_0 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 3/2 & 7/3 \\ -1/2 & -2/3 \end{pmatrix} \quad v_0 = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$$

c) $$A = \begin{pmatrix} -1 & 8 & -2 \\ -2 & 7 & -1 \\ -4 & 10 & 0 \end{pmatrix} \quad v_0 = \begin{pmatrix} 7 \\ 4 \\ 8 \end{pmatrix}$$

15. Consider the matrix

$$A = \begin{pmatrix} -1 & 8 & -2 \\ -2 & 7 & -1 \\ -4 & 10 & 0 \end{pmatrix}$$

of Problem 14(c). For which vectors v does $\|A^k v\| \not\to \infty$ as $k \to \infty$?
[**Hint:** Expand v in an eigenbasis.]

16. **(Fun Example)** The *Fibonacci numbers* are defined recursively as follows:

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n \ \ (n \geq 0).$$

The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ In this problem, you will find a closed formula (as opposed to a recursive formula) for the nth Fibonacci number by solving a difference equation.

a) Let $v_n = \binom{F_{n+1}}{F_n}$, so $v_0 = \binom{1}{0}$, $v_1 = \binom{1}{1}$, etc. Find a state change matrix A such that $v_{n+1} = Av_n$ for all $n \geq 0$.

b) Show that the eigenvalues of A are $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$, with corresponding eigenvectors $w_1 = \binom{-1}{\lambda_2}$ and $w_2 = \binom{-1}{\lambda_1}$.

[**Hint:** Check that $Aw_i = \lambda_i w_i$ using the relations $\lambda_1\lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 1$.]

c) Expand v_0 in this eigenbasis: that is, find x_1, x_2 such that $v_0 = x_1 w_1 + x_2 w_2$. (Don't substitute $\lambda_i = \frac{1}{2}(1 \pm \sqrt{5})$ while doing elimination: express x_1, x_2 in terms of λ_1, λ_2.)

d) Multiply $v_0 = x_1 w_1 + x_2 w_2$ by A^n to show that

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

e) Use this formula to explain why F_{n+1}/F_n approaches the golden ratio when n is large. (Hint: $|\lambda_2| < 1$.)

If you get here and you're thinking "wow, that was cool!", then you might want to consider adding a math major.

17. **(Examples Problem)** Give an example of each of the following, or explain why no example exists.

a) A 3×3 matrix with eigenvalues $1, 2, 3$, and 4.

b) A 2×2 matrix that is not diagonalizable.