Welcome to Math 218D-1!

L1

Introduction to Linear Algebra

What is linear algebra?

→ The study of (systems of) linear equations

Like: $y = 3x + 2$ (rearrange) $-3x + y = 2$

(usually put the variables on the left & constants on the right)

Several equations: $\begin{cases} x + y + z = 1 \\ y - z = 3 \end{cases}$: solve both at once

(arrange in columns to keep things tidy)

Linear means: equations that involve only sums of (number)·(variable) or (number)

Not: $xy + z = 1$ (xy: variable × variable)

$x + 3 = y^2$ (y^2: power of a variable)

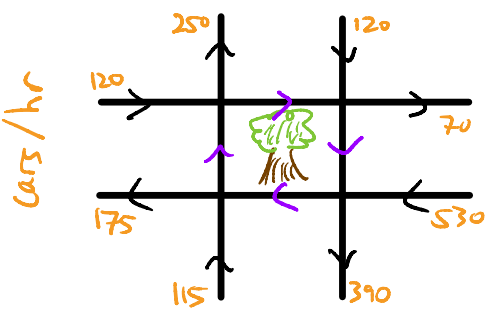
$e^x = \cos(y)$ (e^x, $\cos(y)$: complicated functions)

Why learn linear algebra?

- It's simple enough to understand very well.
- Algorithmic parts can be implemented efficiently on computers.
- It's powerful enough to have a huge range of applications!

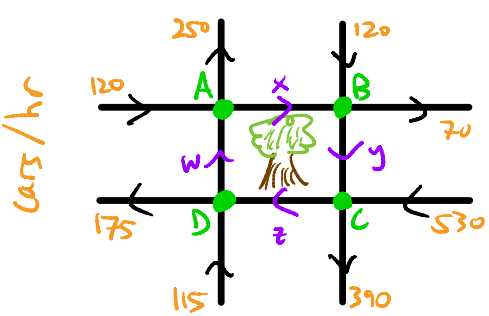
"example"

Eg: Here's a map of roads in the town square:



Question: How many cars/hr travel on the unlabeled roads?

When you have an unknown quantity, name it!



Observation:

cars entering each intersection
= # cars leaving it

A: $120 + w = 250 + x$

B: $120 + x = 70 + y$

C: $530 + y = 390 + z$

D: $115 + z = 175 + w$

rearrange →

variables! ← → constants!

$$\begin{cases} -x + w = 130 \\ x - y = -50 \\ y - z = -140 \\ z - w = 60 \end{cases}$$

columns!

This is a system of 4 linear equations in 4 unknowns.

Question: You know a priori that there are infinitely many solutions. How?

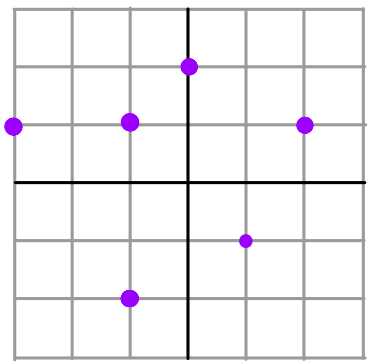
Question: What must be true about the known quantities for a solution to exist?

Linear algebra is a set of tools for solving equations.

Your job is to formulate the problem in terms of linear equations & to interpret the answer in the context of the original problem.

Eg: An asteroid has been observed at coordinates:

$(0,2)$ $(2,1)$ $(1,-1)$ $(-1,-2)$ $(-3,1)$ $(-1,1)$



Question: What is the most likely orbit? Will the asteroid crash into Earth?

Fact (Kepler): The orbit is an **ellipse**.

Equation for an Ellipse:

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

Wait! This equation isn't linear...

If our **data points** lay on the ellipse, then this equation would be satisfied if we substitute the (x,y) values of our data points:

x y

$(0,2)$: $0 + 4B + 0 + 0 + 2E + F = 0$

$(2,1)$: $4 + B + 2C + 2D + E + F = 0$

$(1,-1)$: $1 + B - C + D - E + F = 0$

$(-1,-2)$: $1 + 4B + 2C - D - 2E + F = 0$

$(-3,1)$: $9 + B - 3C + D - 3E + F = 0$

$(-1,1)$: $1 + B - C - D + E + F = 0$

This is a system of 6 linear equations in 5 variables.

"Note"

NB: The variables/unknowns are the coefficients B, C, D, E, F: we're finding the equation of the ellipse!

NB: There is no solution – the points do not lie on an ellipse (measurement error).

Question: What is the best approximate solution?

Answer: the method of least squares (week 8-ish)

[DEMO]

Historical Note: Gauss invented much of the 1st half of this course to (correctly) predict the orbit of the asteroid Ceres in 1801.

Note on Demos: Geometric understanding of linear algebra is a core component in this class.

- The pictures are the material (not a supplement).
- Don't turn off your brain when we do geometry!
- Play with the demos!
- I promise you will have to draw a picture on every exam.

Eg: In a population of rabbits,

(1) Half survive their first year. 

(2) Half of those survive their second year.

(3) Maximum life span is 3 years.

(4) Each rabbit produces (on average)

0, 6, 8 offspring in years

0, 1, 2, respectively.

Problem: Understand the long-term behavior of the rabbit population both qualitatively and quantitatively.

Let's give names to the unknowns.

x_n: # rabbits aged 0 in year n

y_n: # rabbits aged 1 in year n

z_n: # rabbits aged 2 in year n

Our rules say:

$$x_{n+1} = 6y_n + 8z_n$$

$$y_{n+1} = \frac{1}{2}x_n$$

$$z_{n+1} = \frac{1}{2}y_n$$

A system of equations of this form is called a difference equation.

We will solve them using eigenvalues & diagonalization (week 10-ish).

[DEMO] It looks like eventually,

- the population approximately doubles each year.
- the ratio of rabbits aged 0, 1, 2 is $\approx 16:4:1$

We'll learn to extract these facts from the equations!

NB: Google's PageRank algorithm is a special kind of difference equation called a stochastic process. We'll learn about these in week 11-ish.

Eg: Netflix knows what movies you'll like using the Singular Value Decomposition (SVD) and the Principal Component Analysis (PCA). (Weeks 14-15)

The SVD & PCA are the most powerful & most technical tools you will learn this term. In some sense, the whole course is leading up to them.

(I just don't know how to present an example in a few minutes on day 1.)

Geometry of Solutions

Convention: In a system of linear equations, variables go on the left, organized in columns = constants go on the right

$$\begin{cases} -x + w = 130 \\ x - y = -50 \\ y - z = -140 \\ z - w = 60 \end{cases}$$

Def: The solution set of a system of equations is the set of all values for the variables making all equations true simultaneously.

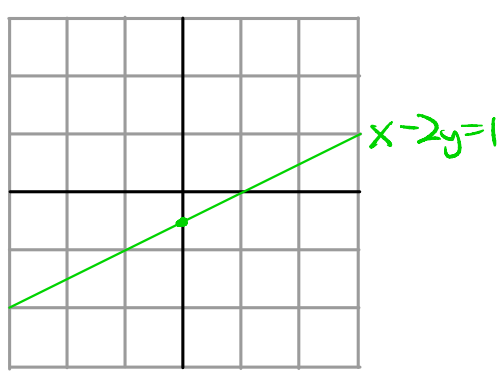
Question: What does the solution set of a system of linear equations look like?

1 equation in 2 variables:

$x - 2y = 1 \rightsquigarrow y = \frac{1}{2}x - \frac{1}{2}$

$y = mx + b$

line in the plane



1 equation in 3 variables:

$x+y+z=1 \rightsquigarrow z=1-x-y$

plane in space

[DEMO]

1 equation in 4 variables:

"3-plane in 4-space"

Note on Dimensions: Is the fourth dimension time? Physicists use 4-space to model spacetime, but you'll probably use it to model other things — like traffic around the town square...

Can you visualize 4-space?

Well, I can't. But I use my pictures in the plane and in 3-space to inform my intuition in higher dimensions.

2 equations in 2 variables:

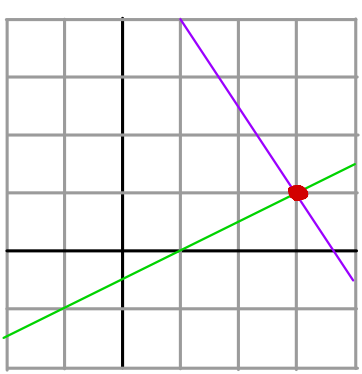
When are both true?

$x-2y=1$

$3x+2y=11$

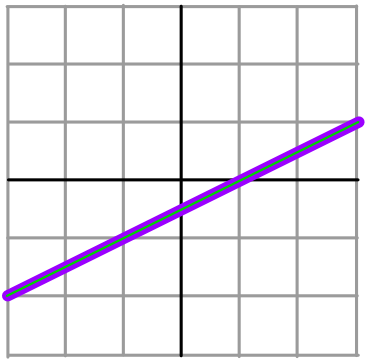
Intersection of 2 lines.

Answer: $(3,1)$



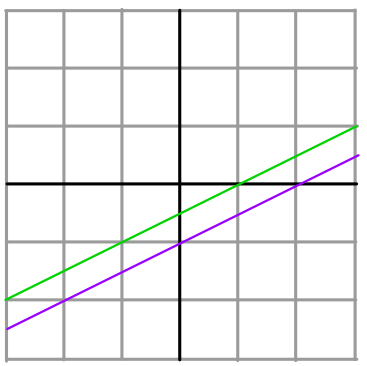
What else can happen?

$x - 2y = 1$
$3x - 6y = 3$



Same line! ∞ solutions.

$x - 2y = 1$
$3x - 6y = 6$



Parallel lines! 0 solutions.

2 equations in 3 variables:

$x + y + z = 1$
$x \quad - z = 0$

[DEMO]

Intersection of 2 planes in space:
in this case, it's a line: ∞ solutions.

3 equations in 3 variables:

$x + y + z = 1$
$x \quad - z = 0$
$y \quad = 0$

Intersection of 3 planes in space: in this case it's a point: one solution.

Question: How many "ways" can 3 planes in space intersect / be arranged?

Answer: I count 8.

Syllabus / Course Information

See the syllabus for details.

- Course materials, calendar, resources, links, etc. are on the course webpage:

  https://services.math.duke.edu/~jdr/2526f-218/index.html

- We will use Canvas for:
  - → Announcements
  - → Gradebook
  - → Gradescope (use school credentials to log in)
  - → Ed Discussion: for asking questions

    > Don't email us with math questions!
    > Post it on Ed — then everyone can see the question & answer.

  - → Warpwire: there are 2 recorded lectures (total). The first is on matrix algebra. Watch it before L2!

Textbook: Officially it's

Strang, "Introduction to Linear Algebra", 5th Ed.

In reality, Joe's Notes are the textbook.

Quizzes:

A very basic, 10-minute small-group quiz will be held at the beginning of each discussion section.

Homework: due Wednesday, 11:59pm every week.

- Scan & submit on Gradescope.
  Use a scanning app.
- Tag the pages on Gradescope with the problems on that page.
- Expect to spend 8-10 hours per week.
- I won't test you on any concept you haven't seen on the homework.
- See the syllabus about late HW tokens
- Graded for correctness.

Midterms: 2 of them, during discussion sections

- Schedule is on the course webpage

Final: as scheduled by the registrar, in lecture room.

ADAPTS: reserve a room at the testing center ASAP for the midterms and the final.

Advice:

- Read the notes! Joe's Notes are the textbook.
- Find a study group!
- Engage with the homework!

  Filling a page with the correct answers doesn't mean you've learned the material.

  Don't shortcut the process! If you think you learned something because ChatGPT's answer makes sense to you, then you're fooling yourself. (Also, using AI on the HW is not allowed.) Try asking real people, like me!
  → More HW tips on Ed
- Don't fall behind!

  The material is highly cumulative and very fast-paced, so it's very difficult to catch up.
- Come to office hours!
- Learning the material is your responsibility.

  If this is your first college math class, you may notice some differences from high school.
  → There's too much material to spoon-feed you every detail.
  → Instead, you'll be drinking from the fire hose.

  Showing up & paying attention in lecture is only the first step to mastery.