L15

Last time: we defined $\det(A)$ using row operations:

(1) If $A \xrightarrow{R_i += c \cdot R_j} B$ then $\det(B) = \det(A)$

(2) If $A \xrightarrow{R_i \times = c} B$ then $\det(B) = c \cdot \det(A)$

(3) If $A \xrightarrow{R_i \leftrightarrow R_j} B$ then $\det(B) = -\det(A)$

(4) $\det(I_n) = 1$.

This is the fastest way to compute the determinant of a general matrix with known entries. However, we are going to care a lot about matrices with unknown entries. Then elimination becomes tedious because you don't know if an entry is a pivot or not.

Eg: $\det\begin{pmatrix} -\lambda & 1 & 3 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = ?$ Is $-\lambda$ a pivot?

Today: Other ways of computing determinants, with an aside on cross products.

Cofactor Expansion

This is a handy recursive formula for det that works well for matrices with unknown entries.

Recursive: we'll compute det(nxn) by computing several det((n-1)x(n-1))'s.

Def: Let A be an $n\times n$ matrix, and $1\le i,j\le n$.

- The (i,j) minor A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row & j^{th} column of A.
- The (i,j) cofactor C_{ij} is

$$C_{ij} = (-1)^{i+j}\det(A_{ij})$$

- The cofactor matrix is the matrix C whose (i,j) entry is C_{ij}:

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots \\ C_{21} & C_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Eg: $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $\qquad A_{21} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$

delete

$$C_{21} = (-1)^{2+1}\det\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} = -(-3) = 3$$

NB: $(-1)^{i+j}$ follows a checkerboard pattern: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$+$: $(-1)^{i+j} = +1$

$-$: $(-1)^{i+j} = -1$

Thm (Cofactor Expansion):

Let A be an $n \times n$ matrix, $a_{ij} = (i,j)$-entry of A.

(1) Cofactor expansion along the i^{th} row:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

(2) Cofactor expansion along the j^{th} column:

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}.$$

Eg: $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(1) Expand cofactors along the 3rd row:

$$\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \underbrace{1}_{a_{31}} \cdot \underbrace{\det\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}}_{C_{31}} + \underbrace{1}_{a_{32}} \cdot \overbrace{(-1)\det\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}}^{C_{32}} + \underbrace{0}_{a_{33}} \cdot \underbrace{\det\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}}_{C_{33}}$$

$$= 1 \cdot (1 \cdot 1 - 2 \cdot 3) - 1 \cdot (0 \cdot 1 - 3 \cdot 1)$$

$$= -5 + 3 = -2$$

(2) Expand cofactors along the 2nd column:

$$\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \underbrace{1}_{a_{12}} \cdot \underbrace{(-1)\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{C_{12}} + \underbrace{2}_{a_{22}} \cdot \overbrace{\det\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}}^{C_{22}} + \underbrace{1}_{a_{32}} \cdot \underbrace{(-1)\det\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}}_{C_{32}}$$

$$= -(-1) + 2(-3) - (-3) = -2$$

Remarks:

(1) This is a recursive formula: $C_{ij} = \pm \det((n-1)\times(n-1))$

(2) You can compute $C_{ij} = (-1)^{i+j}\det(A_{ij})$ however you like — you don't have to use cofactor expansion every time.

(3) Expanding along any row or column gives you $\det(A)$: you always get the same number.

(4) This is ridiculously slow: $O(n!\cdot n)$. It's only useful when your matrix has

→ unknown entries, or

→ a row/column with lots of zeros

Eg: $\det\begin{pmatrix} -\lambda & 1 & 3 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$

$$\underset{\text{1st col}}{\overset{\text{expand}}{=}} (-\lambda)\cdot\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} + 1\cdot(-1)\det\begin{pmatrix} 1 & 3 \\ 1 & -\lambda \end{pmatrix} + 1\cdot\det\begin{pmatrix} 1 & 3 \\ 2-\lambda & 1 \end{pmatrix}$$

$$= -\lambda\big((2-\lambda)(-\lambda)-1\big) - (-\lambda-3) + \big(1-3(2-\lambda)\big)$$

$$= -\lambda(\lambda^2-2\lambda-1) + \lambda+3 + 1-6+3\lambda$$

$$= -\lambda^3+2\lambda^2+5\lambda-2$$

In fact, for 3×3 matrices it's not too hard to compute the determinant even if all of the entries are unknown (like for 2×2).

Eg: $$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\cdot\det\begin{pmatrix} e & f \\ h & i\end{pmatrix} + b(-1)\det\begin{pmatrix} d & f \\ g & i\end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h\end{pmatrix}$$
$$= a(ei-fh) - b(di-fg) + c(dh-eg)$$
$$= aei + bfg + cdh - afh - bdi - ceg$$

Of course, this is only useful if you can remember it.

Sarrus' Scheme:

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix} \qquad \det = aei + bfg + cdh - afh - bdi - ceg$$

This says: to compute det(3×3) matrix:

(1) repeat the first 2 columns on the right

(2) sum the products of the forward diagonals and subtract the products of the backward diagonals

Warning: This only works for 3×3 matrices!

If you try it on a bigger matrix, you won't get the determinant!

→ See the big formula at the end for an $n\times n$ version.

Eg: $$\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 0\cdot2\cdot0 + 1\cdot1\cdot1 + 3\cdot1\cdot1 - 3\cdot2\cdot1 - 0\cdot1\cdot1 - 1\cdot1\cdot0$$

$$\begin{matrix} 0 & 1 & 3 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{matrix}$$

$$= 4 - 6 = \boxed{-2}$$ (again)

Computing 3x3 determinants using Sarrus' scheme vs. cofactor expansion is largely a matter of personal preference.

Here's a matrix with a column with lots of zeros:

Eg: $$\det\begin{pmatrix} 2 & 5 & -3 & -1 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}$$

$$= (-1)(-1)\det\begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} - 5\cdot\det\begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix}$$

$$+ 0\cdot(-1)\det\begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} + 0\cdot\det\begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix}$$

$$= -24 - 5(11) = -79$$

We only had to compute two 3x3 determinants!

Better idea: do a row operation first!

$$\det\begin{pmatrix} 2 & 5 & -3 & -1 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix} \xrightarrow{R_2 -= 5R_1} \det\begin{pmatrix} 2 & 5 & -3 & -1 \\ -12 & -28 & 17 & 0 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}$$

$$= (-1)(-1)\det\begin{pmatrix} -12 & -28 & 17 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} = -79$$

Now we only had to compute one 3x3 determinant!

Summary: Methods for Computing Determinants

(1) Special Formulas (2×2, 3×3)

best for small matrices, although cofactors is still faster for a 3×3 matrix with lots of 0's

(2) Cofactor Expansion

best for matrices with unknown entries, or a row/column with lots of zeros

(3) Row (& column) operations

best if you have a big matrix with known entries & no row/column with lots of zeros

(4) Any Combination of the Above

eg. do a row operation to make a row with lots of zeros, then expand cofactors, using Sarrus' scheme to compute C_{ij}'s ...

Aside: Cofactor Formula for A^{-1}

You can actually compute A^{-1} using cofactors:

Thm: Let C be the cofactor matrix of A. Then

$$AC^T = C^T A = \det(A) \cdot I_n$$

In particular, if $\det(A) \neq 0$ then

$$\boxed{A^{-1} = \frac{1}{\det(A)} C^T}$$

(see the supplement)

NB: This is ridiculously inefficient computationally.

Don't compute A^{-1} this way!

It does tell you that A^{-1} only has $\det(A)$ in the denominators — more on the HW.

One reason I like the cofactor formula is it helps me remember the formula for the 2×2 inverse.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Cross Products

This is an operation that takes two vectors in $\mathbb{R}^3$ and gives you a vector that's orthogonal to both of them.

Recall: The unit coordinate vectors in $\mathbb{R}^3$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Def: Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$. Their cross product is

$$v \times w = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} \in \mathbb{R}^3$$

NB: (vector) $\times$ (vector) = (vector)

but (vector) $\cdot$ (vector) = (scalar)

More than a Mnemonic:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{"}\det \begin{pmatrix} e_1 & e_2 & e_3 \\ \text{---} & v^T & \text{---} \\ \text{---} & w^T & \text{---} \end{pmatrix}\text{"}$$

(interpreted as cofactor expansion along the 1st row)

$$= e_1 \det \begin{pmatrix} b & c \\ y & z \end{pmatrix} - e_2 \det \begin{pmatrix} a & c \\ x & z \end{pmatrix} + e_3 \det \begin{pmatrix} a & b \\ x & y \end{pmatrix}$$

$$= (bz - cy) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - (az - cx) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (ay - bx) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}$$

Eg: $\begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix} = \text{“}\det\begin{pmatrix}e_1 & e_2 & e_3\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}\text{”}$

$$= e_1 \det\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix} - e_2 \det\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + e_3 \det\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$$

$$= -\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

NB: $\begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\-1\end{pmatrix} = 0$ and $\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\-1\end{pmatrix} = 0$ ...

Here's an important relationship between cross products and determinants:

Def: Let $u, v, w \in \mathbb{R}^3$. Their triple product is

$$\boxed{u \cdot (v \times w) = \det\begin{pmatrix} — u^T — \\ — v^T — \\ — w^T — \end{pmatrix}}$$

Check this formula: if $u = \begin{pmatrix}i\\j\\k\end{pmatrix}$, $v = \begin{pmatrix}a\\b\\c\end{pmatrix}$, $w = \begin{pmatrix}x\\y\\z\end{pmatrix}$ then

$$u \cdot (v \times w) = \begin{pmatrix}i\\j\\k\end{pmatrix} \cdot \left[e_1 \det\begin{pmatrix}b & c\\ y & z\end{pmatrix} - e_2 \det\begin{pmatrix}a & c\\ x & z\end{pmatrix} + e_3 \det\begin{pmatrix}a & b\\ x & y\end{pmatrix} \right]$$

$$= \begin{pmatrix}i\\j\\k\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} \det\begin{pmatrix}b & c\\ y & z\end{pmatrix} - \begin{pmatrix}i\\j\\k\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} \det\begin{pmatrix}a & c\\ x & z\end{pmatrix} + \begin{pmatrix}i\\j\\k\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} \det\begin{pmatrix}a & b\\ x & y\end{pmatrix}$$

$$= i \det\begin{pmatrix}b & c\\ y & z\end{pmatrix} - j \det\begin{pmatrix}a & c\\ x & z\end{pmatrix} + k \det\begin{pmatrix}a & b\\ x & y\end{pmatrix}$$

$$= \det\begin{pmatrix}i & j & k\\ a & b & c\\ x & y & z\end{pmatrix}$$

Eg: $u = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\rightsquigarrow v \times w = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (from before)

$\rightsquigarrow u \cdot (v \times w) = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 1 - 3 = -2$

$\Rightarrow \det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = -2$ (again)

Properties of Cross Products:

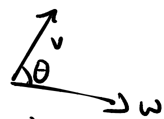
(1) $v \times w \perp v$ and $v \times w \perp w$

$\rightarrow$ because $v \cdot (v \times w) = \det \begin{pmatrix} - & v^T & - \\ - & v^T & - \\ - & w^T & - \end{pmatrix} = 0$

(2) $v \times w = - w \times v$

$\rightarrow$ because "$\det \begin{pmatrix} e_1 & e_2 & e_3 \\ - & v^T & - \\ - & w^T & - \end{pmatrix}$" $\overset{\text{row}}{\underset{\text{swap}}{=}} -$ "$\det \begin{pmatrix} e_1 & e_2 & e_3 \\ - & w^T & - \\ - & v^T & - \end{pmatrix}$"

(3) $\|v \times w\| = \|v\| \cdot \|w\| \cdot \sin(\theta)$

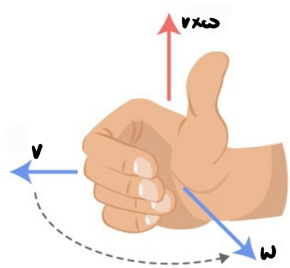


$\rightarrow$ compare $v \cdot w = \|v\| \cdot \|w\| \cdot \cos(\theta)$

(4) $v \times w = 0 \iff v$ and w are collinear

$\rightarrow \sin(\theta) = 0 \iff \theta = 0°$ or $\theta = 180°$

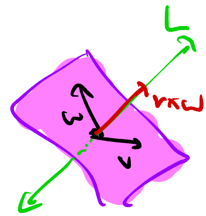
(5) $v \times w$ points in the direction determined by the right hand rule.



NB: (1)+(3)+(5) give a geometric characterization of the cross product:

→ If v, w are collinear then $v \times w = 0$

→ otherwise $\mathrm{Span}\{v, w\}$ is a plane, so $\mathrm{Span}\{v, w\}^\perp$ is a line L.



(1) says $v \times w$ lies on L

(3) says how long $v \times w$ is

(5) says which direction $v \times w$ points.

Why are Cross Products Useful?

They are ubiquitous in multivariable calculus & physics.

For us, it's a shortcut for computing orthogonal complements in $\mathbb{R}^3$.

$$\boxed{\begin{array}{c}\mathrm{Span}\{v, w\}^\perp = \mathrm{Span}\{v \times w\} \\ \text{if } v, w \in \mathbb{R}^3 \text{ are noncollinear}\end{array}}$$

Eg: Find a basis for $\mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}^\perp$

We eyeball $\begin{pmatrix}-2\\1\\0\end{pmatrix} \in \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}^\perp$, but we need one more:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}-3\\-6\\5\end{pmatrix} \rightsquigarrow \text{basis } \left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\-6\\5\end{pmatrix}\right\}$$

The Big Formula (JUST FOR FUNSIES)

This is an explicit, non-recursive formula for $\det(A)$. It's not important for this class, but it's nice to know it exists.

Def: A **permutation** of $\{1,2,\dots,n\}$ is a re-ordering

$$\sigma: \{1,2,\dots,n\} \to \{1,2,\dots,n\}$$

$\sigma(i)$ = new number in the i^{th} position.

So a permutation of $\{1,2,\dots,52\}$ is just a way of shuffling a deck of cards.

Eg: Here are all of the permutations of $\{1,2,3\}$:

123 132 213 231 312 321

i.e. $\sigma(1)=1$ $\sigma(2)=3$ $\sigma(3)=2$

Q: How many permutations of $\{1,2,\dots,n\}$ are there?

- n choices for $\sigma(1)$
- $n-1$ choices for $\sigma(2)$
- $n-2$ choices for $\sigma(3)$

$\vdots$

- 2 choices for $\sigma(n-1)$
- 1 choice for $\sigma(n)$

$n(n-1)(n-2)\cdots 2\cdot 1$

$= n!$ total

NB: $n!$ is a big number! eg. $52! \approx 8 \times 10^{67}$

→ there are only about 10^{24} stars in the universe!

→ so every time you shuffle a deck of cards, you make history – nobody's ever done that before!

→ The Big Formula for a 52×52 matrix has $52!$ summands! It's not computable!

Def: A transposition is a permutation that just swaps two numbers.

Eg: 132 and 213 are transpositions

but 231 is not

Fact: Any permutation can be obtained by doing some number of transpositions.

Eg: $123 \rightsquigarrow 213 \rightsquigarrow 231$ $\quad$ $123 \rightsquigarrow 132 \rightsquigarrow 312$

Def: The sign of a permutation σ is $\text{sign}(\sigma) =$

- $+1$ if σ can be obtained by doing an even number of transpositions
- -1 if σ can be obtained by doing an odd number of transpositions

Eg: $231 = (2 \text{ transpositions}) \Rightarrow \text{sign}(231) = +1$

$132 = (1 \text{ transposition}) \Rightarrow \text{sign}(132) = -1$

Amazingly, this is well-defined — there's no permutation that can be obtained as both an even and an odd number of transpositions.

The Big Formula:

Let A be an $n \times n$ matrix with (i,j) entry a_{ij}.

$$\det(A) = \sum_{\substack{\text{all} \\ \text{permutations} \\ \sigma}} \text{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

In other words, for every way of choosing exactly one entry in each row & column, multiply these entries together, and sum with signs ± 1.

Eg: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

PERMUTATION	#TRANSPOSITIONS	SIGN	SUMMAND
123	0	+1	$a_{11}a_{22}a_{33}$
132	1	−1	$-a_{11}a_{23}a_{32}$
213	1	−1	$-a_{12}a_{21}a_{33}$
231	2	+1	$+a_{12}a_{23}a_{31}$
312	2	+1	$+a_{13}a_{21}a_{32}$
321	1	−1	$-a_{13}a_{22}a_{31}$
			$= \det(A)$