Recall from last time: for a square matrix A,

- an eigenvector is a nonzero vector v such that

$$Av = \lambda v \qquad \lambda = \text{eigenvalue}$$

- The λ-eigenspace is

$$\mathrm{Nul}(A - \lambda I_n) = \{\text{all } \lambda\text{-eigenvectors and } 0\}$$

- The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I_n)$$

  The eigenvalues are the solutions of $p(\lambda) = 0$.

- We like eigenvectors since it's easier to multiply a vector by a scalar than by a matrix, and

$$Av = \lambda v \implies A^k v = \lambda^k v$$

  helps us solve the difference equation

$$v_{k+1} = A v_k \rightsquigarrow v_k = A^k v_0.$$

Today:

- What kind of function is $p(\lambda)$?
- How to solve $v_k = A^k v_0$ if v_0 is not an eigenvector?

The Characteristic Polynomial II

2×2 Matrices: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\det(A - \lambda I_2) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - \overbrace{(a+d)}^{??}\lambda + \underbrace{(ad-bc)}_{\det(A)}$$

This is a polynomial of degree 2.

NB: $p(0) = \det(A - 0 I_2) = \det(A)$, so the constant term of $p(\lambda)$ is always $\det(A)$.

Def: The **trace** of a matrix A is

$\mathrm{Tr}(A)$ = the sum of the diagonal entries of A.

Eg: $\mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$

Characteristic Polynomial of a 2×2 Matrix A

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$$

We can find the zeroes of a degree-2 polynomial using the quadratic formula.

Eg: Find all eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

$\mathrm{Tr}(A) = 4 \quad \det(A) = 3 \quad \rightsquigarrow \quad p(\lambda) = \lambda^2 - 4\lambda + 3 = 0$

$$\Rightarrow \lambda = \frac{1}{2}\left(4 \pm \sqrt{16-12}\right) = \frac{1}{2}(4 \pm 2) = 2 \pm 1$$

So the eigenvalues are 1 and 3.

So what does the characteristic polynomial of a larger matrix look like?

General Form of the Characteristic Polynomial:

If A is an $n \times n$ matrix and $p(\lambda) = \det(A - \lambda I_n)$ then

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \mathrm{Tr}(A) \lambda^{n-1} + (?)\lambda^{n-2} + (?)\lambda^{n-3} + \cdots + (?)\lambda + \det(A)$$

→ This is a polynomial of degree n.

→ You only get the λ^{n-1} and constant coefficients "for free"

→ To compute the other (?) coefficients, you have to compute $\det(A - \lambda I_n)$ (cofactors, ...)

Eg: $A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ In L16 we computed:

$$p(\lambda) = \underbrace{-}_{(-1)^3}\lambda^3 + \underbrace{0}_{\mathrm{Tr}(A)}\lambda^2 + \underbrace{\frac{13}{4}}_{(?)}\lambda + \underbrace{\frac{3}{2}}_{\det(A)}$$

Fact: A polynomial of degree n has at most n roots.

i.e., $p(\lambda) = 0$ has at most n solutions.

Consequence: An $n \times n$ matrix has at most n eigenvalues.

(But each eigenvalue has ∞ eigenvectors.)

How to find the roots of a degree-n polynomial?

- In real life: ask a computer

  NB: The computer will find a matrix whose characteristic polynomial is your polynomial, and use a different, numerically accurate algorithm to find the eigenvals!

- By hand: I won't ask you to find the roots of a polynomial of degree ≥ 3 by hand. (There are lots of fun tricks to do this, though!)

In any case, this is not a Gaussian elimination problem!

Diagonalization

Solving a difference equation is easy when v_0 is an eigenvector:

$$Av_0 = \lambda v_0 \implies v_k = A^k v_0 = \lambda^k v_0.$$

It's also easy when v_0 is a linear combination of eigenvectors. Suppose

$$v_0 = x_1 w_1 + x_2 w_2 + \cdots + x_n w_n \quad \text{where } Aw_i = \lambda_i w_i.$$

Then

$$v_k = A^k v_0 = A^k(x_1 w_1 + x_2 w_2 + \cdots + x_n w_n)$$

$$\xrightarrow{\text{distribute}} x_1 A^k w_1 + x_2 A^k w_2 + \cdots + x_n A^k w_n$$

$$= \underbrace{x_1 \lambda_1^k w_1 + x_2 \lambda_2^k w_2 + \cdots + x_n \lambda_n^k w_n}_{\text{no matrix multiplication! this is computable.}}$$

$$\boxed{\begin{gathered}\text{If } Aw_1 = \lambda_1 w_1,\ Aw_2 = \lambda_2 w_2, \ldots, Aw_n = \lambda_n w_n, \text{ then} \\ A^k(x_1 w_1 + x_2 w_2 + \cdots + x_n w_n) = x_1 \lambda_1^k w_1 + x_2 \lambda_2^k w_2 + \cdots + x_n \lambda_n^k w_n\end{gathered}}$$

Rabbit Example, Cont'd: In L16 we computed the matrix

$$A=\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$ has eigenvalues $\lambda = 2, -\frac{1}{2}, -\frac{3}{2}$.

Compute eigenspaces (bases for $\mathrm{Nul}(A-\lambda I_3)$):

$2: \left\{\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}\right\} \quad -\frac{1}{2}: \left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\} \quad -\frac{3}{2}: \left\{\begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix}\right\}$ (lines)

Let's name these eigenvectors:

$$w_1 = \begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} \quad w_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad w_3 = \begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix}$$

Is our initial state $v_0 = \begin{pmatrix} 16 \\ 6 \\ 1 \end{pmatrix}$ a linear combination of w_1, w_2, w_3? We need to solve

$$\begin{pmatrix} 16 \\ 6 \\ 1 \end{pmatrix} = x_1\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix}$$

$$\xrightarrow[\text{matrix}]{\text{augmented}} \left(\begin{array}{ccc|c} 32 & 2 & 18 & 16 \\ 4 & -1 & -3 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow{\text{solve}} \begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = -1 \end{array}$$

So $v_0 = 1\,w_1 + 1\,w_2 - 1\,w_3$

$$\Longrightarrow v_k = A^k v_0 = 1\cdot 2^k w_1 + 1\cdot\left(-\tfrac{1}{2}\right)^k w_2 - 1\cdot\left(-\tfrac{3}{2}\right)^k w_3$$

If we expand this out, we get

$$v_k = 2^k\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} + \left(-\tfrac{1}{2}\right)^k\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \left(-\tfrac{3}{2}\right)^k\begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k\cdot 32 + (-1/2)^k\cdot 2 - (-3/2)^k\cdot 18 \\ 2^k\cdot 4 + (-1/2)^k\cdot(-1) - (-3/2)^k\cdot(-3) \\ 2^k + (-1/2)^k - (-3/2)^k \end{pmatrix}$$

closed form: no matrix multiplication!

Ok, so we've effectively computed v_k. What about our qualitative observations about v_k as $k \to \infty$?

Observation 1: 2 is the dominant eigenvalue:

"much greater"

$$2^k \gg |(-\tfrac{1}{2})^k| \text{ and } 2^k \gg |(-\tfrac{3}{2})^k| \text{ as } k \to \infty.$$

This means that

$$v_k = 2^k w_1 + (-\tfrac{1}{2})^k w_2 + (-\tfrac{3}{2})^k w_3 \approx \boxed{2^k w_1} \text{ as } k \to \infty$$

(same most significant digits)

This explains why eventually,

- the ratios converge to $(32:4:1)$
- the population roughly doubles every year

Observation 2: $\{w_1, w_2, w_3\}$ is linearly independent

(this is automatic — more on this later)

$$\overset{\text{basis}}{\underset{\text{thm}}{\Longrightarrow}} \{w_1, w_2, w_3\} \text{ is a basis for } \mathbb{R}^3$$

$\Longrightarrow$ any vector in $\mathbb{R}^3$ is a linear combination of w_1, w_2, w_3.

This means any initial state v_0 can be expressed as

$$v_0 = x_1 w_1 + x_2 w_2 + x_3 w_3$$

$$\Longrightarrow v_k = A^k v_0 = 2^k x_1 w_1 + (-\tfrac{1}{2})^k x_2 w_2 + (-\tfrac{3}{2})^k x_3 w_3$$

So Observation 1 holds for any initial state!

Q: What if $x_1 = 0$?

The fact that A has 3 LI eigenvectors means that we can effectively compute $A^k v_0$ for any $v_0 \in \mathbb{R}^3$.

Def: Let A be an $n \times n$ matrix. A is **diagonalizable** if it has n linearly independent eigenvectors $\{w_1, w_2, \ldots, w_n\}$. In this case, $\{w_1, w_2, \ldots, w_n\}$ is called an **eigenbasis**, and computing $\{w_1, w_2, \ldots, w_n\}$ is called **diagonalizing** A.

How to Diagonalize a Matrix:

Let A be an $n \times n$ matrix.

(1) Compute the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I_n)$$

(2) Factor $p(\lambda)$ to find the eigenvalues of A.

(3) Find a basis for each eigenspace.

(4) Combine the vectors from all your bases in (3).

→ If you have n vectors, they form an eigenbasis.

→ Otherwise, A is not diagonalizable.

NB: There's a fact hiding here. Normally if you have several sets of LI vectors and you combine them, it can be LD. Here there's an assertion that if you combine eigenspace bases, you get a LI set. More on this in L18.

Eg: We carried out this procedure for $A=\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$

(1) $p(\lambda) = -\lambda^3 + \frac{13}{4}\lambda + \frac{3}{2}$

(2) The computer says $p(\lambda) = -(\lambda-2)(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})$

(3) We computed a basis for each $\mathrm{Nul}(A-\lambda I_3)$ above:

$$2: \left\{\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}\right\} \qquad -\frac{1}{2}: \left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\} \qquad -\frac{3}{2}: \left\{\begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix}\right\}$$

(4) We got $n=3$ vectors so

$$\left\{\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix}\right\}$$ is an eigenbasis

(automatically LI).

Eg: $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shear

(1) $p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 1$

(2) $\lambda^2 - 2\lambda + 1 = (\lambda-1)^2 \rightsquigarrow$ one eigenvalue 1 (twice?)

(3) Basis for the 1-eigenspace is $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

(4) We only got 1 vector but we needed 2

$\Longrightarrow$ not diagonalizable.

(In L16 we saw that all eigenvectors lie on the x-axis $\Rightarrow$ can't find 2 LI eigenvectors)

NB: The shear should be your favorite example of a non-diagonalizable matrix.

Fact: A "random" matrix will be diagonalizable.

If A is diagonalizable, then any vector in $\mathbb{R}^n$ is a linear combination of eigenvectors.

Def: Let A be a diagonalizable matrix with eigenbasis $\{w_1, w_2, \ldots, w_n\}$ and let $v \in \mathbb{R}^n$. Expanding v in the eigenbasis means solving the vector equation

$$v = x_1 w_1 + x_2 w_2 + \cdots + x_n w_n.$$

or equivalently, the matrix equation

$$\text{this matrix is not } A! \rightarrow \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix} x = v.$$

Important! The point of doing this is that it becomes much easier to multiply by A (or A^k):

$$A^k(x_1 w_1 + x_2 w_2 + \cdots + x_n w_n) = x_1 \lambda_1^k w_1 + x_2 \lambda_2^k w_2 + \cdots + x_n \lambda_n^k w_n.$$

This lets us solve a difference equation $v_k = A^k v_0$ whenever A is diagonalizable.

How to Solve the Difference Equation $v_k = A^k v_0$ when A is Diagonalizable:

(1) Diagonalize A to get an eigenbasis $\{w_1, w_2, \ldots, w_n\}$.

(2) Expand v_0 in the eigenbasis: solve

$$v_0 = x_1 w_1 + x_2 w_2 + \cdots + x_n w_n.$$

(3) Answer: $v_k = A^k v_0 = x_1 \lambda_1^k w_1 + x_2 \lambda_2^k w_2 + \cdots + x_n \lambda_n^k w_n$.

Eg: Solve the difference equation

$$v_{k+1} = Av_k \quad \text{for} \quad A = \begin{pmatrix} 14 & -18 & -33 \\ -12 & 20 & 33 \\ 12 & -18 & -31 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix}.$$

First we diagonalize A:

(1) $$p(\lambda) = \det \begin{pmatrix} 14-\lambda & -18 & -33 \\ -12 & 20-\lambda & 33 \\ 12 & -18 & -31-\lambda \end{pmatrix}$$

expand

$$= (14-\lambda)\det\begin{pmatrix} 20-\lambda & 33 \\ -18 & -31-\lambda \end{pmatrix} - 12(-1)\det\begin{pmatrix} -18 & -33 \\ -18 & -31-\lambda \end{pmatrix}$$

$$+12\det\begin{pmatrix} -18 & -33 \\ 20-\lambda & 33 \end{pmatrix} = \cdots = -\lambda^3 + 3\lambda^2 - 4.$$

(2) Factor:  $\rightsquigarrow -\lambda^3 + 3\lambda^2 - 4 = -(\lambda-2)^2(\lambda+1)$

so the eigenvalues are $\lambda = 2$ and $\lambda = -1$

(twice?)

(3) Find eigenspace bases:

$\lambda = 2$: $$A - 2I_3 = \begin{pmatrix} 12 & -18 & -33 \\ -12 & 18 & 33 \\ 12 & -18 & -33 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & -3/2 & -11/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\text{PVF}}{\rightsquigarrow} x = x_2 \begin{pmatrix} 3/2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 11/4 \\ 0 \\ 1 \end{pmatrix}$$

We can use these vectors as our basis, but let's clear the denominators to make our lives easier:

$$\overset{\text{basis}}{\rightsquigarrow} \left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ 0 \\ 4 \end{pmatrix} \right\}$$

$\lambda = -1$: $$A + I_3 = \begin{pmatrix} 15 & -18 & -33 \\ -12 & 21 & 33 \\ 12 & -18 & -30 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\text{PVF}}{\rightsquigarrow} x = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \overset{\text{basis}}{\rightsquigarrow} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

(4) We ended up with 3 eigenvectors

$$w_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad w_2 = \begin{pmatrix} 11 \\ 0 \\ 4 \end{pmatrix} \quad w_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

So A is diagonalizable and $\{w_1, w_2, w_3\}$ is an eigenbasis.

Now we can solve the difference equation. We expand $v_0 = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix}$ in our eigenbasis:

solve $\begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 11 \\ 0 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$\xrightarrow{\text{augmented matrix}} \left(\begin{array}{ccc|c} 3 & 11 & 1 & 6 \\ 2 & 0 & -1 & 0 \\ 0 & 4 & 1 & 2 \end{array}\right) \xrightarrow{\text{answer}} \begin{aligned} x_1 &= -1 \\ x_2 &= 2 \\ x_3 &= -2 \end{aligned}$$

So $v_0 = -1 w_1 + 2 w_2 - 2 w_3$ and we win:

$$\begin{aligned} v_k = A^k v_0 &= -2^k w_1 + 2^k \cdot 2 w_2 - (-1)^k \cdot 2 w_3 \\ &= 2^k \left[-\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 11 \\ 0 \\ 4 \end{pmatrix} \right] - (-1)^k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= 2^k \begin{pmatrix} 19 \\ -2 \\ 8 \end{pmatrix} - (-1)^k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

$\leftarrow$ closed form: no matrix multiplication.

NB: $2^k \gg |(-1)^k|$ as $k \to \infty$ (2 is the dominant eigenvalue), so $v_k \approx 2^k \begin{pmatrix} 19 \\ -2 \\ 8 \end{pmatrix}$ for large k.

It takes a lot of work to extract this information from the matrix A!

NB: You can't use this procedure to solve $v_k = A^k v_0$ if A is the shear: not diagonalizable! (It's easy to solve by hand)