Complex Eigenvalues

Some matrices don't have any real eigenvalues.
But every matrix has a complex eigenvalue: any polynomial has a complex zero.

Diagonalization works great with complex eigenvalues!
- → You can still solve difference equations.
- → You can still get real-number solutions.

So complex eigenvalues allow us to apply diagonalization techniques to more matrices.

NB: Computers will do this by default! So you have to have some understanding of complex eigenvalues to interpret your computer's output.

Eg: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvalues

(this one was CCW rotation by $90°$: see L16)

$$p(\lambda) = \lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 + 1 = (\lambda + i)(\lambda - i).$$

So the eigenvalues are i and $-i$.

We can compute eigenvectors using the $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ trick:

$$i: \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad -i: \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Check that $A = CDC^{-1}$ for

$$C = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \qquad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In this example, the eigenvalues & eigenvectors were complex conjugates:

$$-i = \bar{i} \qquad \begin{pmatrix} 1 \\ i \end{pmatrix} = \overline{\begin{pmatrix} 1 \\ -i \end{pmatrix}}.$$

This is always the case.

Fact: The complex eigenvalues & eigenvectors of a real matrix come in complex conjugate pairs:

$$Av = \lambda v \iff A\bar{v} = \bar{\lambda}\bar{v}$$

Complex conjugation of vectors is done coordinate-wise:

$$v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \rightsquigarrow \bar{v} = \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix}$$

Eg: Solve the difference equation

$$v_{k+1} = Av_k \qquad A = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 2 \\ 3+\sqrt{3} \end{pmatrix}$$

No complex #s in the statement!

(1) Diagonalize:

$$p(\lambda) = \lambda^2 + 3\lambda + 3 \implies \lambda = \tfrac{1}{2}\left(-3 \pm \sqrt{9-12}\right)$$

eigenvalues: $\lambda = \frac{1}{2}(-3+i\sqrt{3}) \qquad \bar{\lambda} = \frac{1}{2}(-3-i\sqrt{3})$

eigenvectors: $w = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} \qquad \bar{w} = \begin{pmatrix} 1 \\ -\bar{\lambda} \end{pmatrix}$

eigenvector for λ

eigenvector for $\bar{\lambda}$ (complex conj of w)

Check: $Aw = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ 3+3\lambda \end{pmatrix}$

Wait — is this really $= \lambda w$?

$$\lambda w = \lambda \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda^2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \lambda \\ 3+3\lambda \end{pmatrix}$$

$$-\lambda^2 \stackrel{?}{=} 3+3\lambda$$

$$0 \stackrel{?}{=} \lambda^2+3\lambda+3$$

yes — this is $p(\lambda)=0$ ✓

Upshot: it's hard to tell if two complex vectors are scalar multiples just by looking at them!

In any case, $\{w, \bar{w}\}$ is an eigenbasis (eigenvalues are different $\Rightarrow$ they're LI).

(2) Expand v_0 in the eigenbasis:

We need to solve $x_1 w + x_2 \bar{w} = v_0$. Augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ -\lambda & -\bar{\lambda} & 3+\sqrt{3} \end{array}\right) \xrightarrow{R_2 += \lambda R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & \lambda-\bar{\lambda} & 3+\sqrt{3}+2\lambda \end{array}\right)$$

$$\lambda-\bar{\lambda} = \tfrac{1}{2}(-3+i\sqrt{3}) - \tfrac{1}{2}(-3-i\sqrt{3}) = i\sqrt{3}$$

$$3+\sqrt{3}+2\lambda = 3+\sqrt{3}-3+i\sqrt{3} = \sqrt{3}(1+i)$$

$$= \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & i\sqrt{3} & \sqrt{3}(1+i) \end{array}\right) \xrightarrow{R_2 \div= i\sqrt{3}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{1+i}{i} \end{array}\right)$$

$$\frac{1+i}{i} = \frac{1}{i} + 1 = 1-i$$

$$= \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1-i \end{array}\right) \xrightarrow{R_1 -= R_2} \left(\begin{array}{cc|c} 1 & 0 & 1+i \\ 0 & 1 & 1-i \end{array}\right)$$

So $x_1 = 1+i$ and $x_2 = 1-i = \bar{x}_1$, so

$$v_0 = (1+i)w + (1-i)\bar{w}$$

(3) Answer 1:

$$\boxed{v_k = A^k v_0 = \lambda^k (1+i)\, w + \bar{\lambda}^k (1-i)\, \bar{w}}$$

So far this is exactly the same as solving a difference equation with real eigenvalues!

But this isn't a very good answer. We know that

$$v_k = A^k v_0 = (\text{real matrix}) \cdot (\text{real vector})$$

has real coordinates, but our answer has complex numbers all over! How do we eliminate the i's?

Recall: The real part of $z = a+bi$ is

$$\mathrm{Re}(z) = a = \tfrac{1}{2}\big((a+bi)+(a-bi)\big) = \tfrac{1}{2}(z+\bar{z}).$$

(4) Group complex conjugates to get real numbers.

Our two summands in

$$v_k = \lambda^k (1+i)\, w + \bar{\lambda}^k (1-i)\, \bar{w}$$

are complex conjugates:

$$\overline{\lambda^k (1+i) w} = \bar{\lambda}^k\, \overline{(1+i)}\, \bar{w} = \bar{\lambda}^k (1-i)\, \bar{w}.$$

Hence $v_k = \lambda^k (1+i)\, w + \overline{\lambda^k (1+i)\, w} = 2\,\mathrm{Re}\left[\lambda^k (1+i)\, w\right].$

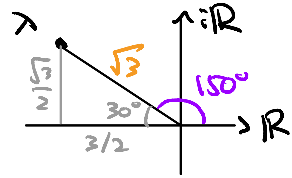
Ok, so how do we compute this?

Recall: z^k is easy to compute if $z = re^{i\theta}$ is in polar form: $z^k = r^k e^{ik\theta}$.

$$\lambda = \frac{1}{2}\left(-3 + i\sqrt{3}\right) = re^{i\theta}$$

$$r = |\lambda| = \frac{1}{2}\sqrt{9+3} = \sqrt{3}$$

$$\theta = 150^\circ = \frac{5\pi}{6}$$



draw a picture!

$$\text{So} \quad \lambda = 3^{1/2} e^{5\pi i/6}$$

$$\Rightarrow \lambda^k = 3^{k/2} e^{5\pi k i/6}$$

$$\text{(Euler's Formula)} = 3^{k/2}\left(\cos\frac{5k\pi}{6} + i\sin\frac{5k\pi}{6}\right)$$

Now we can compute $v_k = 2\,\mathrm{Re}\left[\lambda^k (1+i) w\right]$.

$$\lambda^k (1+i) w$$

$$= 3^{k/2}\left(\cos\frac{5k\pi}{6} + i\sin\frac{5k\pi}{6}\right)(1+i)\begin{pmatrix}1\\ -\lambda\end{pmatrix}$$

multiply

$$= 3^{k/2}\left[\cos\frac{5k\pi}{6} - \sin\frac{5k\pi}{6} + i\left(\cos\frac{5k\pi}{6} + \sin\frac{5k\pi}{6}\right)\right]\begin{pmatrix}1\\ \frac{1}{2}(3-i\sqrt{3})\end{pmatrix}$$

multiply

$$\left[\cos\frac{5k\pi}{6} - \sin\frac{5k\pi}{6} + i\left(\cos\frac{5k\pi}{6} + \sin\frac{5k\pi}{6}\right)\right]\cdot\frac{1}{2}(3-i\sqrt{3})$$

$$= \frac{1}{2}\left(3\cos\frac{5k\pi}{6} - 3\sin\frac{5k\pi}{6} + \sqrt{3}\cos\frac{5k\pi}{6} + \sqrt{3}\sin\frac{5k\pi}{6} + i\,(\text{don't care})\right)$$

$$= 3^{k/2}\begin{pmatrix}\cos\frac{5k\pi}{6} - \sin\frac{5k\pi}{6} + i\,(\text{don't care})\\ \frac{3+\sqrt{3}}{2}\cos\frac{5k\pi}{6} + \frac{\sqrt{3}-3}{2}\sin\frac{5k\pi}{6} + i\,(\text{don't care})\end{pmatrix}$$

$$\Rightarrow v_k = 2\,\mathrm{Re}\left(\text{this}\right) = 3^{k/2}\begin{pmatrix}2\cos\frac{5k\pi}{6} - 2\sin\frac{5k\pi}{6}\\ (3+\sqrt{3})\cos\frac{5k\pi}{6} + (3-\sqrt{3})\sin\frac{5k\pi}{6}\end{pmatrix}$$

Now our answer has only real numbers
(and trig functions — weird)
but we needed complex numbers to get it!

NB: We didn't have to compute the complex part of $\lambda^k(1+i)w$, hence the i (don't care) terms. But we can only do that after multiplying all the complex numbers together!

$$\mathrm{Re}(zw) \neq \mathrm{Re}(z)\mathrm{Re}(w)$$

How to Solve a Difference Equation with $\mathbb{C}$-eigenvalues:

(1–3) Diagonalize A, expand in eigenbasis, solve as before. You'll get something like this:

$$v_k = A^k v_0 = \underbrace{\lambda_i^k x_i w_i + \bar{\lambda}_i^k \bar{x}_i \bar{w}_i}_{\text{complex conjugate pairs}} + \cdots + \underbrace{\lambda_j^k x_j w_j}_{\text{real eigenvalues}} + \cdots$$

(4) Group complex conjugate terms:

$$\lambda^k x w + \bar{\lambda}^k \bar{x} \bar{w} = 2\mathrm{Re}(\lambda^k x w)$$

Write each λ in polar form and use Euler's formula:

$$\lambda = re^{i\theta} \Longrightarrow \lambda^k = r^k e^{ik\theta} = r^k(\cos k\theta + i \sin k\theta)$$

Multiply this by x and w and take the real part.

Now the answer has trig functions instead of i's.

Algebraic and Geometric Multiplicity

Last, we discuss what happens when an $n \times n$ matrix has fewer than n eigenvalues. When is it diagonalizable?

Def: If λ is a zero of a polynomial $p(x)$, its **multiplicity** m is the largest power of $(x-\lambda)$ dividing $p(x)$:

$$p(x) = (x-\lambda)^m \cdot h(x) \qquad h(\lambda) \neq 0$$

Eg: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda-2)^2(\lambda+1)^1$

→ $\lambda = 2$ has multiplicity 2

→ $\lambda = -1$ has multiplicity 1

Def: Let A be a matrix with eigenvalue λ.

(1) The **algebraic multiplicity (AM)** of λ is its multiplicity as a root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$.

(2) The **geometric multiplicity (GM)** of λ is the dimension of the λ-eigenspace:

$$\begin{aligned} GM(\lambda) &= \dim \operatorname{Nul}(A - \lambda I_n) \\ &= \#\text{free variables in } A - \lambda I_n \\ &= \#\text{linearly independent } \lambda\text{-eigenvectors.} \end{aligned}$$

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

The eigenvalues are 1 & 2.

- $\lambda = 1$: $AM = 1$

  $\mathrm{Nul}(A - I_3)$ has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

  This is a line, so $GM = 1$

- $\lambda = 2$: $AM = 2$

  $\mathrm{Nul}(A - 2I_3)$ has basis $\left\{ \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \right\}$

  This is a line, so $GM = 1$

This matrix is not diagonalizable: only 2 LI eigenvectors.

[DEMO]

Eg: $B = \begin{pmatrix} -4 & 3 & 2 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

The eigenvalues are 1 & 2.

- $\lambda = 1$: $AM = 1$

  $\mathrm{Nul}(B - I_3)$ has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

  This is a line, so $GM = 1$

- $\lambda = 2$: $AM = 2$

  $\mathrm{Nul}(B - 2I_3)$ has basis $\left\{ \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$

  This is a plane, so $GM = 2$.

[DEMO]

This matrix is diagonalizable: eigenbasis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$

Both matrices have the same characteristic polynomial $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1 \Rightarrow$ same eigenvalues and AM's.

The difference was that B had $AM(2) = GM(2) = 2$
$\rightsquigarrow$ 2 LI 2-eigenvectors, where A had $GM(2) = 1$
$\rightsquigarrow$ 1 LI 2-eigenvector.

It turns out that these are the only possibilities.

Thm (AM≥GM): For any eigenvalue λ of A,

(algebraic multiplicity of λ)

$\geq$ (geometric multiplicity of λ) ≥ 1

See the supplement for a proof.

NB: $GM \geq 1$ just says that λ has an eigenvector — the λ-eigenspace can't be $\{0\}$, so its dimension is ≥ 1.

Upshot: If $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$ then

- the 1-eigenspace is necessarily a line:

  $1 = AM(1) \geq GM(1) \geq 1 \Rightarrow GM(1) = 1$.

- the 2-eigenspace is a line or a plane:

  $2 = AM(2) \geq GM(2) \geq 1 \Rightarrow GM(2) = 1$ or 2

- the matrix is diagonalizable $\Leftrightarrow GM(2) = 2$: then you have $1+2=3$ LI eigenvectors.

NB: What is not possible is for the 1-eigenspace to be a plane: then $AM(1) = 1 \leq 2 = GM(1)$. So if the matrix is diagonalizable, you have to find 2 eigenvectors with eigenvalue 2.

Thm (AM/GM Criterion for Diagonalizability):
Let A be an $n \times n$ matrix.

- A is diagonalizable over the complex numbers $\iff AM(\lambda) = GM(\lambda)$ for every eigenvalue λ.
- A is diagonalizable over the real numbers $\iff AM(\lambda) = GM(\lambda)$ for every eigenvalue λ and A has all real eigenvalues.

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ is not diagonalizable because $AM(2) = 2 \neq 1 = GM(2)$

Corollary (again): If A has n (different) eigenvalues then A is diagonalizable.

Proof: In this case, $p(\lambda) = \det(A - \lambda I_n)$ factors into distinct linear factors:

$$p(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \qquad \lambda_1, \lambda_2, \ldots, \lambda_n \text{ distinct}$$

so $$1 = AM(\lambda_i) \geq GM(\lambda_i) \geq 1 \implies AM(\lambda_i) = GM(\lambda_i)$$ for every i.

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This is what usually happens: a "random" matrix will have n distinct (complex) eigenvalues.

Eg: Any 2×2 real matrix with a complex eigenvalue is diagonalizable over $\mathbb{C}$: it has 2 eigenvalues $\lambda \neq \bar{\lambda}$.

Proof of the Criterion:

By the fundamental theorem of algebra, the characteristic polynomial factors (over $\mathbb{C}$) into linear terms:

$$p(\lambda) = (-1)^n (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_r)^{m_r} = (-1)^n \lambda^n + \cdots$$

Here $m_i = AM(\lambda_i)$, so $n = \deg(p) = m_1 + m_2 + \cdots + m_r$.

Hence

$$\boxed{n = AM(\lambda_1) + AM(\lambda_2) + \cdots + AM(\lambda_r) \quad \text{(the AMs add to } n)}$$

For each i we have $AM(\lambda_i) \geq GM(\lambda_i)$, so

$$\begin{array}{ccccccccc} n = & AM(\lambda_1) & + & AM(\lambda_2) & + \cdots + & AM(\lambda_r) & & \\ & \mathbin{\rotatebox{90}{$\leq$}} & & \mathbin{\rotatebox{90}{$\leq$}} & & \mathbin{\rotatebox{90}{$\leq$}} & & \\ & GM(\lambda_1) & + & GM(\lambda_2) & + \cdots + & GM(\lambda_r) & = & \text{total \# LI eigenvecs.} \end{array}$$

If any of these is a strict inequality then you have $< n$ LI eigenvectors so A is diagonalizable $\iff AM(\lambda_i) = GM(\lambda_i)$ for each i.

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