Subspaces

... can be hard to motivate.

So far, to every matrix A we have associated **two** linear spaces thru the origin:

(1) Span{cols of A} (= all b vectors making $Ax=b$ consistent)
This is a **parametric description** of a shape.

(2) Solution set of $Ax=0$ (= Span{vectors from PVF})
This is an **implicit description** of a shape.

We want to introduce a notion of "linear space thru 0" called **subspaces** that is **independent** of the particular way we have of describing it.

Another example (L3): both $\begin{aligned} 2x+y+12z&=1 \\ x+2y+9z&=-1 \end{aligned}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$

are **different descriptions** of the **same line!**

We will get a lot of mileage out of changing our description of our subspace to something more convenient for the problem at hand. We want the language to discuss the subspace as its own thing (a shape), without carrying around auxiliary data (numbers).

That said, every subspace can be described as a span:

$\boxed{\text{Subspaces are Spans}}$ and $\boxed{\text{Spans are subspaces}}$

(The same is true if you replace "Span" by "Solution set of a homogeneous system of linear equations".)

So you can think "subspace $= \text{Span}\{???\}$"
(or "subspace = solutions of $(??)x = 0$)
but of course we haven't chosen spanning vectors yet.

Another advantage: we get a criterion for a subset to be a span.

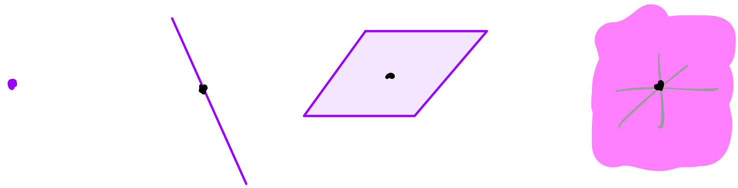
Def: A subset of $\mathbb{R}^n$ is any collection of vectors.

Eg: (a): unit circle (b): first quadrant (c): coordinate axes

$\{(x,y) : x^2+y^2=1\}$ $\{(x,y) : x\geq 0, y\geq 0\}$ $\{(x,y) : xy=0\}$

These are all subsets.

They shouldn't be subspaces though — these don't look like spans, and subspaces look like spans.



Def: A subspace of $\mathbb{R}^n$ is a subset V satisfying:

(1) [closed under $+$] If $u, v \in V$ then $u+v \in V$

(2) [closed under scalar $\times$]
If $v \in V$ and $c \in \mathbb{R}$ then $c \cdot v \in V$.

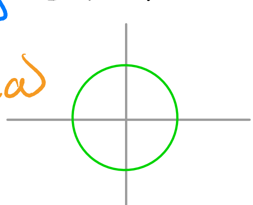
(3) [contains 0] $0 \in V$.

These conditions characterize "linear spaces thru 0" among all subsets.

NB: If V is closed under scalar $\times$ and $v \in V$ then $0 = 0v \in V$ as well, so (3) just forces V to be nonempty: it says $\{\}$ is not a subspace.

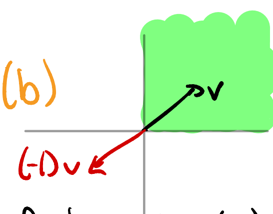
Eg: In the examples above:

(a)



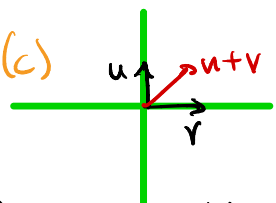
fails (1), (2), (3)

(b)



fails only (2)

(c)



fails only (1)

Here are some easy examples of subspaces.

Eg: $\{0\}$ is a subspace:

(1) if $u, v \in \{0\}$ then $u = v = 0 \Rightarrow u+v = 0 \in \{0\}$ ✓

(2) if $v \in \{0\}$ and $c \in \mathbb{R}$ then $v = 0 \Rightarrow cv = 0 \in \{0\}$ ✓

(3) $0 \in \{0\}$ ✓

NB: $\{0\} = \text{Span}\{\}$ is also a span.

Eg: $V =$ all of $\mathbb{R}^n$ is a subspace

NB: $\mathbb{R}^n = \text{Span}\{e_1, \dots, e_n\}$ is also a span.

The properties are mainly useful for showing that a subset is not a subspace: you just have to find one counterexample to one of the axioms.

whole numbers

Eg: $V = \{(x,y) : x \in \mathbb{Z}, y \in \mathbb{R}\}$ is not a subspace.

It fails (2):

$$\begin{pmatrix}1\\0\end{pmatrix} \in V \quad \text{and} \quad \tfrac{1}{2} \in \mathbb{R} \quad \text{but} \quad \tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \notin V.$$

We will verify that some subsets are subspaces:

Fact: A span is a subspace.

Proof: Let $V = \text{Span}\{v_1, v_2, \dots, v_n\}$.

(1) Let $u, v \in V$. Then there are scalars $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ such that

$$u = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$
$$v = y_1 v_1 + y_2 v_2 + \cdots + y_n v_n.$$

Then

$$u + v = (x_1 v_1 + x_2 v_2 + \cdots + x_n v_n) + (y_1 v_1 + y_2 v_2 + \cdots + y_n v_n)$$
$$= (x_1 + y_1) v_1 + (x_2 + y_2) v_2 + \cdots + (x_n + y_n) v_n \in V.$$

(2) Let $v \in V$ and $c \in \mathbb{R}$. Then there are scalars $x_1, x_2, \dots, x_n$ such that

$$v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

Then
$$\begin{aligned} c \cdot v &= c(x_1 v_1 + x_2 v_2 + \cdots + x_n v_n) \\ &= (cx_1)v_1 + (cx_2)v_2 + \cdots + (cx_n)v_n \in V. \end{aligned}$$

(3) $0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n \in V$ //

Conversely, if V is a subspace and $v_1, v_2, \dots v_n \in V$, $x_1, x_2, \dots, x_n \in \mathbb{R}$ then

$$\begin{aligned} & x_1 v_1, x_2 v_2, \dots, x_n v_n \in V && \text{by (2)} \\ \Rightarrow\ & x_1 v_1 + x_2 v_2 \in V && \text{by (1)} \\ \Rightarrow\ & (x_1 v_1 + x_2 v_2) + x_3 v_3 \in V && \text{by (1)} \\ \Rightarrow \cdots \Rightarrow\ & x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \in V && \text{by (1)} \end{aligned}$$

So any linear combination of $v_1, v_2, \dots, v_n$ is in V

$\Rightarrow$ Span$\{v_1, v_2, \dots, v_n\}$ is contained in V.

Choose enough v_i's to "fill up" V, then:

Fact: A subspace is a span.

More precisely, any subspace can be expressed as a span (in infinitely many ways).

Fact: The solution set of a homogeneous system of linear equations is a subspace.

We know how to use PVF to express the solution set of $Ax=0$ as a span, so this follows from the previous fact, but it is instructive to verify it directly.

Proof: Let $V=$ (solution set of $Ax=0$)

(1) Let $u,v\in V$. Then u,v are solutions of $Ax=0$:

$$Au=0 \quad Av=0$$

$$\Rightarrow A(u+v)=Au+Av=0+0=0$$

Hence $u+v$ is also a solution of $Ax=0$, so $u+v\in V$.

(2) Let $v\in V$ and $c\in\mathbb{R}$. Then v is a solution of $Ax=0$:

$$Av=0 \Rightarrow A(c\cdot v)=c\cdot Av=c\cdot 0=0.$$

Hence $c\cdot v$ is also a solution of $Ax=0$, so $c\cdot v\in V$.

(3) $0\in V$ because $A\cdot 0=0$ (this is the trivial solution).

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Fundamental Subspaces

To any matrix A we will associate four subspaces:

$$\boxed{\mathrm{Col}(A) \qquad \mathrm{Nul}(A) \qquad \mathrm{Row}(A) \qquad \mathrm{Nul}(A^T)}$$

These are just new names for things we have already considered. We'll discuss Col & Nul today.

Def: The column space of a matrix is the span of its columns.

Notation: $\mathrm{Col}(A) = \mathrm{Span}\{\text{Cols of } A\}$

This is a subspace of $\mathbb{R}^m$ $\quad m = \#\text{rows of } A$

$\rightsquigarrow$ column picture $\qquad = \#$ entries in each column

A column space is a span by definition, so we're already verified it is a subspace.

Conversely, $\mathrm{Span}\{v_1, v_2, \ldots, v_n\}$ is the column space of the matrix with columns $v_1, v_2, \ldots, v_n$, so:

Column spaces and Spans are interchangeable.

Eg: $\mathrm{Col}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\right\} \qquad \mathrm{Span}\left\{\begin{pmatrix} 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 9 \\ 10 \end{pmatrix}\right\} = \mathrm{Col}\begin{pmatrix} 7 & 9 \\ 8 & 10 \end{pmatrix}$

If A has columns $v_1, v_2, \dots, v_n$ and $x = (x_1, x_2, \dots, x_n)$ then

$$Ax = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

(using the column-first definition). Hence multiplying by all vectors x gives all linear combinations of the columns:

$$\boxed{\mathrm{Col}(A) = \{ Ax : x \in \mathbb{R}^n \}}$$

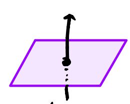
With this description we have another way to see that $b \in \mathrm{Col}(A) \iff$ there is an x such that $Ax = b$, ie, $\iff Ax = b$ is consistent.

Column Picture Criterion for Consistency, revisited:

$Ax = b$ is consistent $\iff b \in \mathrm{Col}(A)$

(infinitely) Many different matrices will have the same column space.

Eg: $\mathrm{Col}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = $ (the xy-plane)



$$= \mathrm{Col}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Col}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \cdots$$

(any 2 noncollinear vectors in the xy-plane span it)

So there are many different descriptions of a subspace as a Col space.

Def: The null space of a matrix A is the solution set of $Ax=0$.

Notation: $\text{Nul}(A) = \{x : Ax=0\}$

This is a subspace of $\mathbb{R}^n$ $n = \#\text{columns}$
$= \#\text{variables}$

$\rightsquigarrow$ row picture

We verified already that $\text{Nul}(A)$ is a subspace.

Try not to be intimidated by the new terminology:

A null space is just the solution set of a homogeneous system of equations

(infinitely)
Many different matrices will have the same null space.

Indeed, row operations do not change the solution set!

Eg: $\text{Nul}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \overset{R_2 -= 4R_1}{=} \text{Nul}\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix} \overset{R_2 \div= -3}{=} \text{Nul}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ etc.

So there are many different descriptions of a subspace as a Nul space.

We know how to find a spanning set for $\text{Nul}(A)$:

$$\mathrm{Nul}(A) \quad \boxed{\xrightarrow[\text{(Gauss–Jordan)}]{\text{PVF}}} \quad \mathrm{Span}\left\{ \begin{array}{c}\text{vectors}\\ \text{from PVF}\end{array} \right\}$$

WORK

Eg: Express $\mathrm{Nul}\begin{pmatrix}1&2&2&1\\2&4&1&-1\end{pmatrix}$ as a span/Col space.

We have to find the solution set of

$$\begin{pmatrix}1&2&2&1\\2&4&1&-1\end{pmatrix}x=0$$

in PVF.

$$\begin{pmatrix}1&2&2&1\\2&4&1&-1\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}1&2&0&-1\\0&0&1&1\end{pmatrix}$$

$$\rightsquigarrow \begin{aligned} x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= - x_4 \\ x_4 &= x_4 \end{aligned} \quad \rightsquigarrow \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

So $\mathrm{Nul}\begin{pmatrix}1&2&2&1\\2&4&1&-1\end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\right\} = \mathrm{Col}\begin{pmatrix}-2&1\\1&0\\0&-1\\0&1\end{pmatrix}$

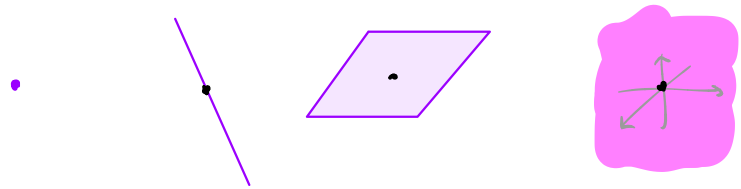
(As with $\mathrm{Col}(A)$, there are infinitely many other ways to do this.)

This procedure lets us pass between two descriptions of a subspace: Nul $\rightsquigarrow$ Span / Col.

Perspective:

Col & Nul are two ways of describing a subspace using matrices (so a computer can do computations).

But when you hear subspace you should think shape:



You should not think numbers:

$$\mathrm{Col}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \qquad \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{pmatrix}$$

Implicit vs Parametric Descriptions

Col(A) is a span, so it is a parametric description:

$$\mathrm{Col}\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix} = \text{all vectors of the form } \overbrace{x_1}^{\text{parameters}}\begin{pmatrix}1\\2\\3\end{pmatrix} + \overbrace{x_2}\begin{pmatrix}4\\5\\6\end{pmatrix}$$

→ It is easy to produce vectors in Col(A):

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4\\5\\6\end{pmatrix},\quad 2\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}4\\5\\6\end{pmatrix},\ \ldots \in \mathrm{Col}\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$$

→ It is hard to check if a vector is in Col(A):

$$\begin{pmatrix}7\\8\\9\end{pmatrix}\in \mathrm{Col}(A)\,? \rightsquigarrow \text{solve } \begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}x=\begin{pmatrix}7\\8\\9\end{pmatrix}\ldots$$

Nul(A) is a solution set, so it is an implicit description:

$$\mathrm{Nul}\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix} = \left\{\text{solutions of } \overbrace{\begin{matrix}x_1+2x_2+3x_3=0\\4x_1+5x_2+6x_3=0\end{matrix}}^{\text{equations}}\right\}$$

→ It is easy to check if a vector is in Nul(A):

$$\begin{pmatrix}1\\-2\\1\end{pmatrix}\in \mathrm{Nul}\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}? \qquad \begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}\overset{?}{=}\begin{pmatrix}0\\0\end{pmatrix} \checkmark$$

→ It is hard to produce vectors in Nul(A):

$$\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}x=0 \underset{\text{(work)}}{\overset{\text{PVF}}{\rightsquigarrow}} \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=x_3\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

In practice, if someone hands you a subset V, either:

(a) You will find a counterexample to one of the subspace axioms to show that V is not a subspace, or:

(b) You will express V as a column or null space

→ this verifies that V is a subspace

→ now you can do computations on V.

In particular, you will (basically) never verify that a subset is a subspace using the 3 axioms.

Doing (b) is more of an art than a science, but one rule of thumb is:

- Is V defined in terms of parameters?
  Is it easy to produce vectors in V?
  → probably easier to express as a col space/span.

- Is V defined in terms of equations?
  Is it easy to check if a vector is in V?
  → probably easier to express as a null space.

Eg: $V = \{(x,y,z) \in \mathbb{R}^3 : x+y=z\}$

This is defined by an equation, so it's probably a null space. In fact, $x+y=z \iff x+y-z=0$

so $V = \mathrm{Nul}\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$.

Eg: $V = \left\{ \begin{pmatrix} 3a+b \\ a-b \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$

This is defined by parameters. It's easy to produce vectors in V: eg. $a=1, b=2 \rightsquigarrow \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \in V$.

So it's probably a column space. Rewrite:

$$\begin{pmatrix} 3a+b \\ a-b \\ b \end{pmatrix} = a\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\longrightarrow V = \mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} = \mathrm{Col}\begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}.$$