Linear (In)Dependence

L7

Last time: We defined subspaces = linear spaces thru 0, and various ways of describing them:

(1) As a span / Col space — parametric

(2) As a solution set / Nul space — implicit

Today we focus on (1). We'll see that

⭐ Some spanning sets are better than others. ⭐

In particular, you might be using too many vectors!

Eg: $V = \mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ is a plane:

The first two vectors are noncollinear $\Rightarrow$ they span a plane, and the last is a LC of the first 2:

$$\begin{pmatrix}-1\\5\\12\end{pmatrix} = \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

[DEMO]

so the 3 vectors are coplanar. In particular,

$$\mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$$

This reduces the number of parameters used to produce the vectors in the plane:

$$x_1\begin{pmatrix}2\\-4\\6\end{pmatrix} + x_2\begin{pmatrix}2\\-5\\1\end{pmatrix} + x_3\begin{pmatrix}-1\\5\\12\end{pmatrix} \quad \text{vs.} \quad x_1\begin{pmatrix}2\\-4\\6\end{pmatrix} + x_2\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

(You should only need 2 numbers to sweep out a plane.) Moreover, the parameterization with 2 parameters is unique, but with 3 parameters it is redundant:

$2\left(-\frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix}+3\begin{pmatrix}2\\-5\\1\end{pmatrix}+\begin{pmatrix}-1\\5\\12\end{pmatrix}\right)=0$

[DEMO]

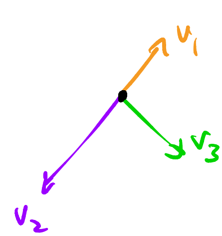
$$\begin{pmatrix}0\\1\\5\end{pmatrix} = 1\begin{pmatrix}2\\-4\\6\end{pmatrix} - 1\begin{pmatrix}2\\-5\\1\end{pmatrix} + 0\begin{pmatrix}-1\\5\\12\end{pmatrix} = -4\begin{pmatrix}2\\-4\\6\end{pmatrix} + 5\begin{pmatrix}2\\-5\\1\end{pmatrix} + 2\begin{pmatrix}-1\\5\\12\end{pmatrix}$$

but $\begin{pmatrix}0\\1\\5\end{pmatrix} = x_1\begin{pmatrix}2\\-4\\6\end{pmatrix} + x_2\begin{pmatrix}2\\-5\\1\end{pmatrix}$ only for $x_1=1$, $x_2=-1$.

We want to formalize the notion that there are "too many" spanning vectors by saying one vector is in the span of the others.

In the example above, any 2 noncollinear vectors span a plane, and no pair of spanning vectors was collinear, so any one of those vectors is in the span of the others. This is not always the case.

Eg: $v_1=\begin{pmatrix}1\\1\end{pmatrix}$ $v_2=\begin{pmatrix}-2\\-2\end{pmatrix}$ $v_3=\begin{pmatrix}1\\-1\end{pmatrix}$



Here $v_2=-2v_1+0v_3 \in \mathrm{Span}\{v_1,v_3\}$

but $v_3 \notin \mathrm{Span}\{v_1,v_2\} =$ (line)

We want a condition that means some vector is in the span of the others, without specifying which.

Answer: rewrite as a homogeneous vector equation:

$$v_2 = -2v_1 + 0v_3 \rightsquigarrow -2v_1 - v_2 + 0v_3 = 0.$$

Def: A list of vectors $\{v_1, v_2, \dots, v_n\}$ is **linearly dependent (LD)** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$$

has a *nontrivial* solution $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$.
Such a solution is called a **linear relation**.

move here

Eg: $$\begin{pmatrix}-1\\5\\12\end{pmatrix} = \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix} \Longrightarrow \underbrace{-\begin{pmatrix}-1\\5\\12\end{pmatrix} + \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix} = 0}_{\text{linear relation}}$$

$$\Longrightarrow \left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\} \text{ is LD}$$

move here

Eg: $$v_2 = -2v_1 + 0v_3 \Longrightarrow \underbrace{-2v_1 - v_2 + 0v_3 = 0}_{\text{linear relation}}$$

$$\Longrightarrow \{v_1, v_2, v_3\} \text{ is LD.}$$

NB: If $x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$ with some $x_i \neq 0$ then

$$v_i = -\frac{1}{x_i}(x_1v_1 + x_2v_2 + \cdots + x_{i-1}v_{i-1} + x_{i+1}v_{i+1} + \cdots + x_nv_n)$$
$$\in \operatorname{Span}\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$$

This explains why:

$\{v_1, \dots, v_n\}$ is LD $\iff$ some v_i is in the span of the others

Recall: $Ax = 0$ has a nontrivial solution $\iff$ A has a free variable. (Otherwise the only solution is 0.)

So $\{v_1, \dots, v_n\}$ is LD

$\iff x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$ has a nontrivial soln

$\iff$ the matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ has a free variable.

This is how you can check if vectors are LD (and find a linear relation).

Eg: Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ LD? If so, find a linear relation.

We have to solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

free $\Rightarrow$ LD

$$\rightsquigarrow \underset{\text{matrix}}{\text{augmented}} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{\text{PVF}}{\rightsquigarrow} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Taking any $x_3 \neq 0$ gives a nonzero soln $\rightsquigarrow$ linear relation:

$$x_3 = 1 \qquad 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

Eg: Any set containing the zero vector is LD:

$0v_1 + 0v_2 + 1\cdot 0 = 0 \implies \{v_1, v_2, 0\}$ is LD

Eg: If $r > n$ then any r vectors in $\mathbb{R}^n$ are LD:

$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{pmatrix}$ is a wide matrix $\Rightarrow$ has a free variable

eg: any 3 vectors in $\mathbb{R}^2$ are necessarily LD:

$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ has a free variable. [DEMO]

NB: "$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}5\\6\end{pmatrix}\right\}$ is LD" means "a nonzero soln of $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} x = 0$ exists." A linear relation means "here is an actual nonzero solution: $(x_1, x_2, x_3) = (\dots)$".

Summary: Let $v_1, v_2, \dots, v_n$ be vectors.

The Following Are Equivalent (TFAE):

for given vectors, either they're all true or they're all false

(1) $\{v_1, v_2, \dots, v_n\}$ is linearly dependent

(2) The matrix

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

has a free variable.

(3) Some v_i is in the span of the others.

Vectors that are not LD are called:

Def: A list of vectors $\{v_1, v_2, \dots, v_n\}$ is **linearly independent (LI)** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has only the trivial solution $(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$.

A logically equivalent definition is:

$\{v_1, v_2, \dots, v_n\}$ is LI if $\quad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$ implies $x_1 = x_2 = \cdots = x_n = 0$

The logical negation of the Summary above is:

Summary: Let $v_1, v_2, \dots, v_n$ be vectors.
The Following Are Equivalent:

(1) $\{v_1, v_2, \dots, v_n\}$ is linearly independent

(2) The matrix

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

has a pivot in every column

(3) No v_i is in the span of the others.

Roughly, (3) means vectors are LI if their span is as large as it can be: every time you add a vector, the span gets bigger!

Linguistic Note: According to the definition, LI/LD are adjectives that apply to a list of vectors.

Bad: "A is LI"
Good: "A has LI columns"

Bad: "v_1 is LD on v_2 and v_3"
Good: "$\{v_1, v_2, v_3\}$ is LD" or "$v_1 \in \mathrm{Span}\{v_2, v_3\}$"

Eg: Is $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\-8\\9\end{pmatrix}\right\}$ LI or LD?

We have to solve the vector equation

$$x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} + x_3\begin{pmatrix}7\\-8\\9\end{pmatrix} = 0$$

$\rightsquigarrow$ augmented matrix $\begin{pmatrix}1 & 4 & 7\\2 & 5 & -8\\3 & 6 & 9\end{pmatrix} \overset{\text{ref}}{\rightsquigarrow} \begin{pmatrix}1 & 4 & 7\\0 & -3 & -22\\0 & 0 & 32\end{pmatrix}$

There's a pivot in every column $\Rightarrow$ no nontrivial solution $\Rightarrow$ LI. Or, if you prefer:

$$\left(\begin{array}{ccc|c}1 & 4 & 7 & 0\\2 & 5 & -8 & 0\\3 & 6 & 9 & 0\end{array}\right) \overset{\text{rref}}{\rightsquigarrow} \left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right) \Rightarrow \begin{array}{l}x_1=0\\x_2=0\\x_3=0\end{array}$$

[DEMO]

LI means Parameterizations are not Redundant:

If $\{v_1, v_2, \ldots, v_n\}$ is LI and $b \in \mathrm{Span}\{v_1, v_2, \ldots, v_n\}$ then there are unique weights $x_1, x_2, \ldots, x_n$ such that

$$b = x_1v_1 + x_2v_2 + \cdots + x_nv_n$$

Proof: The associated matrix equation

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} x = b$$

is consistent $(b \in \operatorname{Span}\{v_1, v_2, \ldots, v_n\}) \Rightarrow$ has 1 soln (no free variables). //

What do LI/LD mean in geometric terms?

Eg: $\{0\}$ is LD.

Eg: $\{v_1\}$ is LD $\Longleftrightarrow v=0$

so $\{v_1\}$ is LI $\Longleftrightarrow \operatorname{Span}\{v_1\}$ is a line (dim 1)

[DEMO]

Eg: $\{v_1, v_2\}$ is LD $\Longleftrightarrow v_1 \in \operatorname{Span}\{v_2\}$ or $v_2 \in \operatorname{Span}\{v_1\}$

ie v_1 & v_2 are collinear

[DEMO]

so $\{v_1, v_2\}$ is LI $\Longleftrightarrow \operatorname{Span}\{v_1, v_2\}$ is a plane (dim 2)

Eg: $\{v_1, v_2, v_3\}$ is LD $\Longleftrightarrow v_1 \in \operatorname{Span}\{v_2, v_3\}$ or $v_2 \in \operatorname{Span}\{v_1, v_3\}$

or $v_3 \in \operatorname{Span}\{v_1, v_2\}$

ie. v_1, v_2, v_3 are coplanar

so $\{v_1, v_2, v_3\}$ is LI $\Longleftrightarrow \operatorname{Span}\{v_1, v_2, v_3\}$ is space (dim 3)

et cetera...

[DEMO]

$$\boxed{\text{The span of } n \text{ LI vectors has dimension } n}$$

Basis and Dimension

The $\boxed{\text{red box}}$ above says we should define the dimension of a subspace V by

(1) describing V as a span of LI vectors

(2) counting how many vectors we used.

Step (1) is the notion of a basis.

Def: Let V be a subspace. A set of vectors $v_1, v_2, \ldots, v_n \in V$ is a **basis** for V if:

(1) $V = \mathrm{Span}\{v_1, v_2, \ldots, v_n\}$

(2) $\{v_1, v_2, \ldots, v_n\}$ is LI.

In other words, a basis gives a way to describe V as a span of LI vectors.

NB: A basis is a minimal spanning set. LI $\Rightarrow$ if you remove any vector from your basis, the span gets smaller!

NB: Spans + LI $\Rightarrow$ we get a unique parameterization of the vectors in V: every $b \in V$ can be written as

$$b = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

for unique values $x_1, x_2, \ldots, x_n$.

We want to use "Span{n LI vectors} has dim n" to define dimension now, but first we need to know:

Fact: Any basis for V has the same number of vectors.

In other words, you can't have different descriptions

$$V = \mathrm{Span}\{v_1, v_2\} \qquad \text{and} \qquad V = \mathrm{Span}\{w_1, w_2, w_3\}$$

with $\{v_1, v_2\}$ and $\{w_1, w_2, w_3\}$ both LI.

Def: The **dimension** of a subspace V is the number of vectors in any basis.

Notn: $\dim(V)$

We'll see below why this agrees with our Provisional Defn in L3.

In fact, once you know $\dim(V)$, if you have the right # vectors then it's a basis if it spans or it's LI.

Basis Thm: Suppose $\dim(V) = d$. Let $v_1, v_2, \ldots, v_d \in V$.

- If $\mathrm{Span}\{v_1, v_2, \ldots, v_d\} = V$ then $\{v_1, v_2, \ldots, v_d\}$ is LI (so it's a basis for V)
- If $\{v_1, v_2, \ldots, v_d\}$ is LI then $\mathrm{Span}\{v_1, v_2, \ldots, v_d\} = V$ (so it's a basis for V)

Eg: (1) Any 2 vectors that span a plane are LI.
(2) Any 2 LI vectors in a plane form a basis.

Eg: The only basis for the point $\{0\}$ is $\{\}$ $\quad$ ($\{0\}$ is LD)
$\Rightarrow \dim\{0\} = 0$ ✓
(A point had better have a basis with zero vectors!)

Eg: Any nonzero vector in a line V is a basis
$\Rightarrow \dim V = 1$ ✓

Eg: Any two noncollinear vectors in a plane V form a basis
$\Rightarrow \dim V = 2$ ✓

etc.

Eg: Find a basis for $V = \operatorname{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$

This is a plane, so we just need 2 non-collinear vectors. For instance,

$$\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\} \text{ or } \left\{\begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\} \text{ or } \left\{\begin{pmatrix}2\\-5\\1\end{pmatrix}, \underbrace{\begin{pmatrix}1\\0\\13\end{pmatrix}}_{\begin{pmatrix}2\\-5\\1\end{pmatrix}+\begin{pmatrix}-1\\5\\12\end{pmatrix}}\right\}$$

Eg: $\{e_1, e_2, \ldots, e_n\}$ forms a basis for $\mathbb{R}^n$.

Indeed,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{= e_1} + x_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{= e_2} + \cdots + x_n \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{= e_n}$$

so they span $\mathbb{R}^n$. LI: if $\uparrow = 0$ then all $x_i = 0$.

Hence $\boxed{\dim \mathbb{R}^n = n}$ — our notion of dimension passes this sanity check.

NB: The only n-dimensional subspace of $\mathbb{R}^n$ is all of $\mathbb{R}^n$. Indeed, if V is in $\mathbb{R}^n$ and $\dim V = n$ then V contains n LI vectors (a basis). But the Basis Thm $\Rightarrow$ any n LI vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$.

NB: $\mathbb{R}^n$ has (infinitely) many bases!

Eg: Any 2 noncollinear vectors form a basis for $\mathbb{R}^2$ (a plane):

$$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \quad \text{or} \quad \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} \quad \text{or} \quad \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\} \quad \cdots$$

In fact, any nonzero subspace has infinitely many bases — they are highly non-unique.

→ A basis is some way of describing a subspace using the fewest parameters.

Warning! Be careful to distinguish these 3 things:

Subspace	Basis	Matrix
$V = \mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$	$\begin{pmatrix}2 & 2\\-4 & -5\\6 & 1\end{pmatrix}$
This is a plane. It has ∞ vectors in it.	This is a basis for V. It has 2 vectors in it.	This is a matrix. Its columns form a basis for V.

To pass from a basis to a subspace you take the span:

$$\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\} \rightsquigarrow \mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\} = \mathrm{Col}\begin{pmatrix}2 & 2\\-4 & -5\\6 & 1\end{pmatrix}$$

How do you find a basis for a subspace?

Recall: To do computations on a subspace, first express it as $V = \mathrm{Col}(A)$ or $V = \mathrm{Nul}(A)$.

Basis for Col(A)

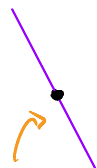
Thm: The pivot columns of A form a basis of $\mathrm{Col}(A)$.

$$\begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix} \overset{\text{ref}}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \text{basis: } \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

pivot column

NB: You have to take the pivot columns of the original matrix, not its REF! This is because row operations change the column space:

$$\mathrm{Col}\begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} =$$



different!

$$\text{but } \mathrm{Col}\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} =$$

NB: This procedure produces one of infinitely many bases for Col(A)! Don't become too attached to one basis — there may be a more convenient one to use.

(later: orthogonal bases...)

Proof: Let R be the RREF of A. Eg:

$$A = \begin{pmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{pmatrix} \rightsquigarrow R = \begin{pmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns

Note $Ax = 0 \iff Rx = 0$ (same solution sets). This says the cols of A & cols of R satisfy the same linear relations.

Spans: To show that the pivot columns span $\mathrm{Col}(A)$, we need to show that the other columns are already in the span of the pivot columns. This is certainly true for R: for example,

$$\underset{\text{last col}}{\begin{pmatrix} 4 \\ 6 \\ -1 \\ 0 \end{pmatrix}} = 4 \underset{\text{1st col}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} + 6 \underset{\text{2nd col}}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} - \underset{\text{4th col}}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}$$

Rewrite: $$4\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \\ -1 \\ 0 \end{pmatrix} = 0$$

This is a linear relation among the cols of R, so the cols of A satisfy the same relation:

$$4v_1 + 6v_2 - v_4 - v_5 = 0 \implies v_5 = 4v_1 + 6v_2 - v_4$$

$$\implies v_5 \in \mathrm{Span}\{v_1, v_2, v_4\} \quad \checkmark$$

LI: Suppose $x_1v_1 + x_2v_2 + x_4v_4 = 0$. The cols of R satisfy the same relation:

$$0 = x_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_4\\0\end{pmatrix}$$

This implies $x_1 = x_2 = x_4 = 0$, so $\{v_1, v_2, v_4\}$ is LI. ✓

Since the pivot cols of A span Col(A) & are LI, they form a basis. //

Eg: Find a basis for $V = \text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$.

This is the same as $\text{Col}\begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix}$. Let's find the pivot columns:

$$\begin{pmatrix}2 & 2 & -1\\-4 & -5 & 5\\6 & 1 & 12\end{pmatrix} \overset{\text{ref}}{\rightsquigarrow} \begin{pmatrix}2 & 2 & -1\\0 & -1 & 3\\0 & 0 & 0\end{pmatrix}$$

So $\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$ is the basis produced by this procedure.

Def: The rank of A is $\text{rank}(A) = \#\text{ pivots of } A$.

Consequence: The number of vectors in any basis for Col(A) equals the number of pivots:

$$\boxed{\dim \text{Col}(A) = \#\text{ pivots of } A = \text{rank}(A).}$$

Basis for Nul(A)

Thm: The vectors attached to the free variables in the parametric vector form of the solns of $Ax=0$ form a basis for Nul(A).

Eg: Find a basis for $\mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\leadsto \begin{aligned} x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned} \leadsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\leadsto \text{basis } \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

NB: As before, this produces one of infinitely many bases.

Proof: Every solution of $Ax=0$ has the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

(that was the point of PVF).

So these vectors span Nul(A). We have to check that they are LI.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies x_2 = x_4 = 0 \quad //$$

Consequence: The number of vectors in any basis for $\mathrm{Nul}(A)$ is equal to the number of free variables:

$$\boxed{\dim \mathrm{Nul}(A) = \#\text{free variables} = \#\text{columns} - \mathrm{rank}(A)}$$

NB: Before (L3) we provisionally defined the dimension of the solution set of a consistent system $Ax=b$ to be the # free variables. But we know

$$\begin{pmatrix} \text{solns of} \\ Ax=b \end{pmatrix} = \begin{pmatrix} \text{particular} \\ \text{solution} \end{pmatrix} + \mathrm{Nul}(A)$$

and $\dim \mathrm{Nul}(A) = \#$ free variables, so this is consistent with the definition in this lecture.